Michele Emmer
Editor

Imagine Math 3

Between Culture and Mathematics

Springer

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Contents

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Introduction

Michele Emmer

Imagine all the people
Sharing all the world
John Lennon

Imagine mathematics, imagine with the help of mathematics, imagine new worlds, new geometries, new forms. Imagine building mathematical models that make it possible to manage our world better, imagine solving great problems, imagine new problems never before thought of, imagine music, art, poetry, literature, architecture, theatre and cinema with mathematics. Imagine the unpredictable and sometimes irrational applications of mathematics in all areas of human endeavour.

Imagination and mathematics, imagination and culture, culture and mathematics. For some years now the world of mathematics has penetrated deeply into human culture, perhaps more deeply than ever before, even more than in the Renaissance. In theatre, stories of mathematicians are staged; in cinema Oscars are won by films about mathematicians; all over the world museums and science centres dedicated to mathematics are multiplying. Journals have been founded for relationships between mathematics and contemporary art and architecture. Exhibitions are mounted to present mathematics, art and mathematics, and images related to the world of mathematics.

This third volume of the series begins with the connections between art and mathematics including the papers of L. D. Henderson, E. Barisoni, U. Bottazzini, M. Rottman and M. Emmer. All the papers are connected to the large exhibition on Art and Math entitled "Visible Harmonies" that was to take place at the MART in Italy during 2013, and was cancelled a few weeks before the opening. An article by J. P. Bourgignon is dedicated to the exhibition he organized in Paris in 2013 on math and modern Art. The volume includes an homage to Benoit Mandelbrot by N. Samuel and G. A. Losa. Very interesting the connections between math,

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architecture and design are addressed in the papers by G. Legendre, E. Fiorelli and others, and A. Giuliani.

An interesting part is dedicated to the connections between math, computer science and theatre with the papers by C. Bardainne and A. Mondot.

All topics are treated in a rigorous but captivating way, detailed but full of evocations. An all-embracing look at the world of mathematics and culture.

Part I
Mathematics and Art

Visual Harmonies

Michele Emmer

> Mathematics has determined the direction and content of much philosophic thought, has destroyed and rebuilt religious doctrines, has supplied substance to economic and political theories, has fashioned major painting, musical, architectural and literary styles, has fathered our logic . . . Finally, as an incomparably fine human achievement mathematics offers satisfactions and aesthetic values at least equal to those offered by any other branch of our culture.

So wrote the math historian Morris Kline in the book *Mathematics in Western Culture* [3, p. 150].

In 1623, Galileo wrote in *The Assayer* (*Il Saggiatore*):

> I seem to discern the firm belief that in philosophizing one must support oneself upon the opinion of some celebrated author, as if our minds ought to remain completely sterile and barren unless wedded to the reasoning of some other person. Possibly he thinks that philosophy is a book of fiction by some writer, like the *Iliad* or *Orlando Furioso*, productions in which the least important thing is whether what is written there is true. Well, Sarsi, that is not how matters stand. Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth [2, pp. 237–338].

Thus, without mathematical structures we cannot understand nature. Mathematics is the language of nature.

A few centuries later, in 1904, a famous painter wrote to Emile Bernard:

> Traiter la nature par le cylindre, la sphère, le cône, le tous mis en perspective, soit que chaque côte d'un objet, d'un plan, se dirige vers un point central. Les lignes parallèles à l'horizon donnent l'étendue, soit une section de la nature. Les lignes perpendiculaires à cet horizon donnent la profondeur. Or, la nature, pour nous hommes, est plus en profondeur qu'en surface, d'ou la nécessite d'introduire dans nos vibrations de lumière, représentée par les rouges et le jaunes, une somme de blues, pour faire sentir l'air [6, pp. 268–269]

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> (Treat nature by the cylinder, the sphere, the cone, all put into perspective, that is, each side of an object, of a plane, is directed towards a central point. The lines parallel to the horizon give the extension, or a section of nature. The lines perpendicular to this horizon give depth. But nature, for us, is more depth than surface, whence the necessity of introducing into our vibrations of light, represented by red and yellow, a certain amount of blue, to feel the air.)

The art historian Lionello Venturi commented that in the paintings by Cezanne (the famous artist in question) there are no cylinders, spheres and cones, so the artist's quote represents nothing but an ideal aspiration to an organization of shapes transcending nature.

> It is easy to see how mathematics has contributed to changing our concept of space—the space in which we live and the idea of space itself. This is because mathematics is not merely a means of measurement in recipes, but has contributed, if not determined, the way in which we understand space on earth and in the universe.
>
> These are some of the reasons behind a major exhibition dedicated to the idea of space and the relationships between modern and contemporary art and mathematics from the twentieth century to the present day at the MART in Rovereto, one of the most important European museums of modern and contemporary art. The exhibition entitled 'Visible harmonies: the idea of space between art and mathematics', will probably open March 2013 and will remain open until the end of June 2013.
>
> The general idea of the exhibition is by Michele Emmer, the project by Gabriella Belli, director of the MART, and Michele Emmer, and the organisation by Eisabetta Barisoni. The other members of the scientific committee are: Umberto Bottazzini, Università di Milano; Linda D. Henderson, University of Texas at Austin; and Michael Rottman, Frei Universität Berlin. The curators of the catalogue, in two different volumes, Italian and English, are Gabriella Belli and Michele Emmer [1, p. 121–122].

This is what I wrote in 2011 about the exhibit.

Today, in 2013, I know that this exhibition will never come to be. The preparatory work went on for 4 years, the scientific committee met about ten times in Rovereto to organise the design of the exhibition, the catalogue, and the list of works. Contact had been made with the university and the MUSE, the museum of science in Trento, designed by Renzo Piano, which opened on 27 July 2013, for collateral initiatives: lectures, exhibitions, films, theatrical productions. By the end of 2011 the catalogue was almost complete, a substantial number of works had already been obtained through loans from other museums around the world. In short, the exhibition was in its final stages. Then several things happened that were not foreseen. The director of the MART accepted a very prestigious position, to preside over all the museums in the city of Venice, from the Museo Correr, to the Ducal Palace, to the museum of Ca' Pesaro, to the glass museum in Murano, and others. The transfer took place at the beginning of 2012. A new director was nominated for the MART. In consequence of the ongoing financial crisis in Italy, even a wealthy region such as Trentino-Alto Adige has had its funding cut, and these impact the MART as well. As early as October 2011 I saw clearly that the exhibition would never open. It is not easy to accept that 4 years of work will not produce any concrete result, especially when you've arrived towards the end of your scientific career, and that exhibition

represented the culmination of 40 years of work. Finally, after much insistence, in February 2012 the official announcement arrived that the project had been cancelled.

I am still convinced that it was the right moment, as has been shown by the 2012 exhibition in Paris that is discussed in this present volume, and as shown by many, constantly increasing signs of interest in the ties between mathematics and art. Not the least of these is the initiative of one of the most important journals for art, science and technology, published by MIT Press: *Leonardo* (on whose editorial board I have served for 20 years). In collaboration with the National Science Foundation of the United States, Leonardo launched a program named 'SEAD *Network for Sciences, Engineering, Arts and Design* [5] It began with a white paper presented at the Smithsonian Institution in Washington D.C. on 16 May 2013, and the project is going forward:

> SEAD will share results and seek feedback from an exploration of challenges and opportunities for transdisciplinary research and creative work, informed by 200 international contributors. Next we will consider methods for innovative exchanges supporting cross-disciplinary learning across formal and informal education settings. Partner group SEAD will then discuss designs for a 21st-century online portal to references and displays of work resulting from transdisciplinary collaboration [4].

I thought that the work of so many fine minds shouldn't go wasted, and thus I asked all those who were part of the scientific committee for the exhibition to participate in the congress in Venice in 2013. That is why there a section in this present volume that has the same name that the exhibition was to have had.

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6. L. Venturi, *La via dell'impressionismo: da Manet a Cezanne* (Einaudi, Torino, 1970), pp. 268–269

From Modern to Eternal: Art and Mathematics in the Return to Order of the 1920s

Elisabetta Barisoni

> The great art of the Mediterranean, of which Italian art is a part, ideally emulates architecture in each of its branches; abandoning the analytical imitation and reproduction of reality ... it aims at creating the atypical synthesis of it. From the modern it re-ascends to the eternal, and from chance to definitive ([1], p. 101; my trans.).

Dal moderno all'eterno (From modern to eternal) is the title of an essay that Margherita Sarfatti considered writing at the beginning of the 1920s. It was never actually written but perhaps converged with her later writings on art criticism, *Segni, colori e luci* [2] and *Storia della pittura moderna* [1930].[1] Sarfatti is a central figure for those dealing with the period between the two world wars, not only because of her intense activities as a journalist, writer, cultural protagonist and promoter of the artistic circle of Novecento Italiano, but also and above all because she was a tenacious and attentive communicator of the ideas that circulated in Italy and Europe. Beginning with her theoretical starting points, it is possible to retrace some of her thinking about mathematics and the return to order through the works of some of the best known artists who worked with this poetic (Fig. 1).

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[1] In [19], the critic wrote: *Un volume di critica e storia artistica, e di riflessioni intorno all'arte,* Dal moderno all'eterno*, è da tempo fra le carte della mia cartella e fra i pensieri cari e accarezzati dal mio spirito* (A volume of criticism and art history, and of reflections about art, *From modern to eternal,* has been among the papers in my files, and in the thoughts most dear to and cherished by my spirit). Margherita Grassini was born in Venice in 1880, to a wealthy Jewish family. After her marriage to the lawyer Cesare Sarfatti and their move to Milan, her work as an art critic intensified, and she came increasingly closer to the definition of a consistent and complex aesthetic vision. As is known, the threads of Sarfatti's personal and professional lives were interwoven with socialist thought, and then with the newborn Fascist movement; her contribution to the political development of Benito Mussolini remains an open historiographical question. Research concerning Sarfatti only began with the studies of Rossana Bossaglia and Claudia Gian Ferrari [20]. For her work as an art critic, see [21] and [22]. However, much still remains to be investigated, and the present author is currently involved in the study of the Sarfatti archive acquired by the Museo di Arte Moderna e Contemporanea di Trento e Rovereto (MART) (MART, Archivio del '900, Fondo Margherita Sarfatti), comprised of materials that are almost completely unpublished and which makes it possible to study in depth Sarfatti's work as a publicist, art critic and promoter of Italian culture in the period 1919–1939.

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Fig. 1 Margherita Sarfatti with beret and necklace, portrait by Ghitta Carell [ca. 1925–1930]. *Source*: MART, Archivio del '900, Fondo Margherita Sarfatti, Sar. 5.1.1.26

In the 1920s, order implied a continual reference to tradition,[2] or 'modern tradition' as Sarfatti loved to define it. The legacy of classical work was flanked by the contemporary recovery of idealist philosophy—above all Platonic philosophy—and was inserted into an altered cultural panorama, aimed at the search for rigour, solidity, construction and sometimes mathematisation of reality. Here it is not possible to reconstruct the complex developments of idealism, or better, of twentieth-century idealism, but it is still useful to note that concepts such as geometry, number as an ordering principle, harmony, equilibrium and, to use Sarfatti's terminology, precision of sign, determination, and synthesis, became part of the common vocabulary of the expressions and practices of an entire generation of artists.[3]

2 Among the numerous studies and publications on the return to order, see the fundamental exhibition *Les Realismes entre révolution et réaction, 1919–1939* [23]. For bibliographical references see also [24, 25] and the most recent exhibitions [26, 27].

3 Sarfatti's conception of synthesis was one of the cardinal points of her critical theory. For example, she wrote, *Ma l'epoca dell'analisi è finita; la guerra ne ha segnato il tramonto, con la simbolica e allegorica sconfitta del popolo analizzatore per eccellenza, la Germania. L'ora della sintesi è suonata* (But the epoch of analysis is finished: the war marked its decline, with the symbolical and allegorical defeat of the analyzing people par excellence, Germany. The hour of synthesis has arrived) ([1], p. 143). For the symbolist origins of the concept of synthesis, see the famous essay by [23].

'We Walk Towards Classicism'

> Besides, nothing is closer to art than mathematics, my sweet friend. … Times and things have changed and tend to change even more. A great desire for order, for precision and for clarity ignites all souls, and an epoch lies at our feet like molten bronze that waits to be cast in the form of a colossal statue. We walk towards classicism (A. Soffici to Callistene Agonista (Giovanni Papini), Paris, 20 February 1903, quoted in ([3])

Soffici's letter of 1903 reminds us that the return to order was not the product of the aftermath of the first world war, but emerged in the decade that preceded it, in the full flower of the avant-garde. The long trajectory of art 'from the modern to the eternal' first of all involved the conception of time, as Massimo Bontempelli writes in an editorial in the first issue of the periodical *900*: 'The most pressing task of the twentieth century will be the reconstruction of time and space' ([4]). It was a notion of time in which all ages are contemporary, or better, according to another definition dear to Sarfatti, a time in which there was a perfect equilibrium between modernity and eternal tradition. The new art aspired to overcome the fragmentariness of impressionist poetics, as did the movement of contemporary life that the avant-garde sought to represent.[4] The shift began with Paul Cézanne, who had turned his attention from impressionism to the problem of representing reality, 'solidifying it', as has been said. In Sarfattian thought the master painter from Aix-en-Provence is the true *trait-d'union* between the nineteenth century and the poetic that emerged towards the end of the 1910s. This 'Frenchman from the south, of half-Italian blood' led towards the 'eternal climate of art, beyond all contingency and superficiality … this was the torment and tragedy of Paul Cézanne. To arrive at the still, the composed, the ordered and the "metaphysical" definitive'.

In this train of thought, each canvas is a step towards great art, 'it is another victory of the spirit over matter, the achievement of a beauty that is at once more profound and more pared down, more atypical, more elevated'.[5] This is how Sarfatti expressed it in 1924, writing about Novecento Italiano painters: 'Gods long in exile, here now the master-ideas, the general ideas; here now "the concepts" return to the domain of the plastic arts'.[6] The subject of art is no longer the mutational

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4 See [1], p. 24, where she states that impressionism 'negates the stable and definitive, this highest aspiration of ideal and classical art' (my trans.).

5 See ([1], p. 28 and p. 31, and 146–147): *Precisione nel segno, decisione nel colore; risolutezza nella forma; sentimento profondo e sobrio, scavato e scarnito attraverso la meditazione, l'eliminazione e lo studio; aspirazione verso il concreto, il semplice e il definitivo; ecco i tratti comuni—l'aria di famiglia—di questa generazione di artisti* (Precision in sign, decisiveness in colour; resolution in form; profound and sober sentiment, hollowed out and stripped of flesh by means of meditation, elimination and study; an aspiration to the concrete, simple and definitive; these are the common traits—the family resemblance—of this generation of artists). These concepts are constants in Margherita Sarfatti's thought; see also [29]. There is a vast body of critical literature concerning Cézanne and his success during the 1910s and 1920s; here we need only note that in 1920 the Venice Biennale dedicated an entire room to his work alone.

6 MART, Archivio del'900, Fondo Margherita Sarfatti, Sar 3.3.28 *Giudizii su 6 pittori del 9cento*, draft manuscript. In 1924 Sarfatti published the article "Mostra di 'Sei pittori del'900'" in the catalogue of the 14th Venice Biennale. Some parts of the manuscript do not agree with the text published in the

quality of becoming, more than once defined by Bontempelli as raw material, needy of antibiotic treatment, but the world of ideas (see [5]). Here again is Sarfatti's description of the line as the 'sublime effort of our idealism to purify into its essence the innumerable random expressions of form' ([1], p. 15).

Platonic thought was already spreading throughout the Cubist milieu in France; in 1916 Amédée Ozenfant published an extract of Plato's *Philebus* [6], a text he had been introduced to by Léonce Rosenberg, who was the celebrated owner of the gallery L'Effort Moderne as well as a scholar of philosophy. Philebus was mentioned again in the third article published in *Il Popolo d'Italia* that Sarfatti dedicated to the 1919 Futurist exposition in Milan [7], the Grande Esposizione Nazionale Futurista then taking place at the Galleria Centrale d'Arte [see also ([1], p. 48)]. To introduce the paintings of Sironi, Funi and Dudreville, Sarfatti wrote:

> 'The beauty of figures which I am now trying to describe is not what most people would understand as such, not the beauty of a living creature of a picture; what I mean, what the argument points to, is something straight, or round, and the surfaces and solids which a lathe, or a carpenter's rule and square, produces from the straight and the round. I wonder if you understand. Things like that, I maintain, are beautiful now, like most things, in a relative sense; they are always beautiful in their very nature, and they carry pleasures peculiar to themselves which are quite unlike the pleasures of scratching. And there are colours too which have this characteristic' [Plato, Philebus, 51c-d] [8]. These words are not from Picasso, nor from Gleizes or Metzinger, the two theorists of Cubism. Rather, they date back several millennia; Plato put them in the mouth of Socrates. Paul Cézanne certainly did not know them, but listen to how he seems to comment on them: 'Everything in nature is modelled on the lines of the sphere, the cone and the cylinder, and one must learn to paint these: he will then know how to paint all things no matter what they are'. The new generation of painters and sculptors, the new art that evolved and re-elaborated by their efforts, cannot be understood without drawing on this cornerstone, the great art of all time, whose utmost effort and utmost expression is concretised in a plastic intellectuality, diverse in form and accent, but one in essence: the deformation and recomposition of reality, according to the volitive and constructive unity of style ([7], p. 3).

Through Cézanne was inaugurated the line of Sarfattian 'modern classicism', a classicism permeated with the Platonic idealism that had animated, centuries before, the work of masters such as Piero della Francesca, Leonardo and Raphael. This was the route of form reconstruction that also necessarily had to be followed by the group of Novecento Italiano artists and by Italian art along with them, at least as its most active promoter saw it.

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catalogue, in particular the phrase *ecco ora "i concetti"* (here now are "the concepts") appears only in the manuscript.

From Cubism to Classicism: Art and Mathematics in Gino Severini

The numerous references to Plato that circulated in those years in the milieu of Parisian late-Cubism and beyond were often drawn from on the late dialogues, those traditionally closer to Pythagorean thought. Pythagoras is another leading figure in the return to order, and was above all taken up again thanks to considerations found in Renaissance treatises, from Leon Battista Alberti to Luca Pacioli.[7]

As early as 1918 in *Après le cubisme* Ozenfant and Jeanneret, the future Le Corbusier, had this to say: 'Number, which is the basis of all beauty, has found its expression ... The proven laws are human constructions that coincide with the order of nature and can be represented by numbers' ([9]).

Margherita Sarfatti, always up to date about European trends, and French thinking above all, grasped the innovative importance of the theories and practices of purism, such as that of Le Corbusier and Perret, but also warned of the hazard of an art that was cold, excessively rational, Nordic, the opposite of the one she preferred, which was classical-Mediterranean, synthetic, permeated by a vivid humanism. In light of this, we can understand the tepid welcome that Sarfatti gave one of the texts that is fundamental for the study of the 'mathematical' climate of the 1920s, *Du cubisme au classicisme. Esthètique du compas et du nombre* published in Paris by Gino Severini [10].

Already in 1916, at a very early date, with the exception of Picasso's parallel returns to classicism, Severini anticipated the themes of *Du Cubisme* in creating works such as *Maternity* which, stripped of all autobiographical and dynamic notes, concentrates on the archetypical ideal of maternity, timeless, without any connotations, movement, 'intrusions' of a given reality, as Sarfatti would say, an eternal representation. The classical pose reprises that of the Virgins of the 1300s and 1400s, but to the Italian visual tradition Severini adds the studies that he was making of Plato, Pythagoras and the neo-Platonists,[8] drawing from these sources the rigorous construction of the woman with child, which comes out inscribed in a 'pure' geometrical figure like the triangle. Even though Piero is mentioned only once in *Du cubisme*, Fagiolo dell'Arco rightly defines *Maternity* as 'a painting that is geometrical, elementary, but also luminous and of simplified colour: a Tuscan,

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[7] According to Pythagorean philosophy, everything in nature is number, and the divine cause itself had used a numerical ratio to create the universe, conferring on it a peculiar character of harmony and equilibrium. Identifying the constitutive formulas of reality, the ratios that bind things, in short, that number underlying all things visible, signifies following the divine gesture at the moment of creation. Not by chance was this ratio called by Luca Pacioli *divine proportion*, by Leonardo 'golden section', by Kepler *sectio divina*. It is the golden number, God's secret cipher, a constant that must be identified and reprised by the artist.

[8] Pontiggia, in his Italian edition of *Du cubisme* [30], cites, among others, Theon of Smyrna's *Mathematics Useful for Understanding Plato*, a work from the age of Hadrian that was translated into French in 1892.

Piero della Francescan painting' ([11], p. 37; my trans.).[9] Severini's way of working in the 1920s was also influenced by a 'classical' elaborated management, as Daniela Fonti notes:

> The execution of each painting is preceded by a series of studies on paper that determine the composition according to complex 'tracings' based on rotations of geometric figures, symmetries and correspondences. Now Severini is aiming at discovering and reducing to norms that are practical and of universal validity, the harmonic laws that govern the works of art of the past, which he strives to emulate very precociously (1916, *Maternity*; *Portrait of Jeanne*). In his still lives (*Still Life,* 1920) he orchestrates a spatially rich and complex depiction, from separate and distinct points of view, that is implacably logical and at the same time almost visionary: a sort of Cézanne revisited according to 'the aesthetic of the compass and number' ([12], pp. 26–27).[10]

Even while sharing his love for the classical world, and for the genius of Leonardo, whose words act as the introductions for some of the chapters of Severini's book, and even while acknowledging the high quality of the painter from Cortona's rendering of 'modern classicism', Sarfatti finds his *Du Cubisme* too dogmatic. The reasons for this lie in the 'Mediterranean' humanism mentioned earlier, of which the art critic never ceased to promote:

> Life, ah me!, is man! This is what is most difficult to re-encapsulate in the geometry of shape and of abstract formula. ... This renovated classicism, which now comes out of limbo, in order to become true classicism must surpass the milieu of the object, of the thing, and become human again. In this sense humanism was classical, because the classical measure par excellence remains man ([13]).[11]

[9] See also [31].

[10] [32] also contains the most recent bibliography for Severini. Further stimulation arrived to Severini by the coeval work of Paul Sérusier who published his *ABC de la peinture* in 1921, and from the reflections of George Seurat and Maurice Denis, who published his *Théories 1890-1910: Du symbolisme et de Gauguin vers un nouvel ordre classique* in 1912. On the theory of colour see also ([33], pp. 153–154), cited in ([32], pp. 176–177).

[11] On 'humanism' see also [1], p. 135: *Questa eresia di logica estrema, paradossale e puritana; questo eccesso di ragione pura sino all'irragionevole, è di sua natura nordico, cioè fanatico e vorrei dire protestante. Risponde al materialismo cerebrale e al raziocinio astratto della geometria e della matematica applicate ai fenomeni dello spirito, che è complesso, delicato, duttile* (This extreme, paradoxical and puritan heresy of logic; this excess of pure reason up to the irrational, is part of its Nordic character, that is, fanatic and I would say Protestant. It responds to cerebral materialism and abstract reasoning of geometry and mathematics applied to phenomena of the spirit, which is complex, delicate, ductile). For Sarfatti's judgment of *Du Cubisme* see [34]. As regards Leonardo, it is useful to recall that his *Trattato* had been translated into French in 1909 and circulated in the Parisian artistic milieu.

Carlo Carrà and Felice Casorati

The reprisal of Platonic thought and the call for formal construction are the basis of the poetics of many artists with varying degrees of closeness to Novecento Italiano but whose works are exhibited alongside them, in Italy and abroad.

Noteworthy among these is Carlo Carrà, who already in the mid-1910s turns his gaze, without pause, from Futurism to metaphysics and the recovery of the classical tradition, enriching it with strong Platonic accents. The works of this period testify to an intensification of the presence of pyramids, set squares, cubes, spheres and spheroids, not due to a need for a purely formal or linguistic renovation, but as a consequence of an altered philosophical horizon.[12] In his writings the artist constantly refers to concepts such as the 'harmonious proportions' and the 'remote rules and measures', showing a retracing of the Platonic tradition which had been travelled from the Medieval to the modern age. The references are to Piero della Francesca's *De prospectiva pingendi,* to Luca Pacioli's *De divina proportione* and to Albrecht Dürer's *De symmetria partium in rectis formis humanorum corporum*. In the same years, close to the activity of the group named Valori Plastici, Carrà studied Giotto and Paolo Uccello; further, in his autobiographical writing of 1918, *Il ritorno di Tobia*, he calls himself 'a new poet, even though in those times his faculties were turned to matheamtics and to the strangest and most curious volumes of antique erudition' ([14], p. 185).[13] Thus he writes in his *Pittura metafisica* of 1919, the year in which he created his painting-manifesto *Lot's Daughters:* 'Since nature is the force to which art gives action and form, only the *Platonic idea* of reality can translate into a work of fantasy a clear form of reality' ([15], vol. 1; [16]). Giotto, Paolo Uccello, Leonardo, Raphael and above all Piero della Francesca are the cultural references for an entire generation. They are almost 'obligations' for the artists, considered that, as Flavio Fergonzi underlines, Carrà himself, in *La mia vita* of 1943, appends the reference to Piero a posteriori:

> It was towards the end of 1915 when I took up again with renewed passion the aesthetic problems inherent in plastic transcendence, and the expressions concretised through the work of Paolo Uccello and Piero della Francesca made a valid contribution to my understanding of spatiality as Euclidean geometry and the cornerstone of the constructive values and the *golden section*, used in the composition and division of spaces ([17], p. 232).[14]

[12] Monferini writes, *Attraverso il manichino, Carrà vuole invece rifarsi a un'idea semplificata e astratta del corpo umano come elemento geometrizzante in un'armonia di accennati rapporti matematici e spaziali* (By means of the mannequin, Carrà wishes to remake a simplified and abstract idea of the human body as a geometricizing element in a harmony of accented mathematical and spatial relationships) ([35], p. 87). Carrà's Platonism has been widely discussed; instead, often neglected is how close the artist was to Margherita Sarfatti and the fact that he carried on an uninterrupted epistolary relationship with her until her decline and exile.

[13] For more by Carrà, see (C. [36–38]). The artist also published a monograph on [39].

[14] See also [40], where she notes that Carrà himself followed the example of the *Giotto maestro* of Aix-en-Provence and had already in 1919, in *La Ronda*, proclaimed modernity.

Among the most ‘Piero della Francescan’ artists was Felice Casorati, like Carrà a participant in the undertakings of Novecento Italiano art, although in an independent position. Among the masterpieces of the period, *Portrait of Silvana Cenni* (1922) depicts the student who animated *Lo studio* (1923), another fundamental work lost in 1931 in the fire that destroyed the Glaspalast in Munich where it was on exhibit, along with the third painting of this ideal triptych, *La lezione* (1929).

> The immediate iconographic reference for *Silvana Cenni* is precisely the Piero della Francesca of the *Polyptych of the Misericordia*, a work that is emblematic for that reflection on the mathematics alluded to, seen as the principal divine regulator of reality and of the work of art. *Silvana Cenni* is a hieratic and solemn figure, on the ancient model, in an interior made of silences and meditation, where geometry seems to order everything, from the ceiling to the human body, even the shadows. But the rigour is only apparent, the symmetries subtly offset, and the light comes from two different sources in the room, meeting on the neck and face of the woman. Even in the forced perspective the references to the art of the past are continuous, it is sufficient to think of the reflections on perspective of artists such as Paolo Uccello.[15]

This is how Fagiolo dell’Arco describes it:

> An absurd geometry, which twists even the lineaments to make them resemble un equation, which exploits every micro-element (such as the roll on the left) to go more deeply into the spatial theme. The extraneous perfection of the egg, the logic of the ‘functional line’ and the ‘tactile values’, but with the addition of the notation (subtly Germanic) of the *Malinconia* ([11], p. 39 ff.).

Mario Sironi, Architect of Melancholy

Jean Clair speaks precisely of melancholy with regard to the work of Mario Sironi, one of the leading figures of Novecento Italiano, and one of the greatest interpreters of Sarfattian critical thought.

Like the ‘constructions’ of Severini, the works-manifestos *The Architect* (1922) and *The Student* (1924) are ideal portraits, mental operations. The solid figures, the geometric objects and forms, the definite architectonic settings are not derived from reality but are instead born of a strong constructive will which, realised through line, orders the compositive rigour. ‘To construct’ appears to be one of the keywords of these years.[16] Sironi seems to be exactly the one who incarnates the aspect of solidity and at the same time essentiality, ‘the flesher of reality’ dear to Sarfatti. In a text of 1931 she describes it this way:

[15] Translated from the entry on *Silvana Cenni* compiled by the present author, in ([41], pp. 134–135).

[16] In *Pittura metafisica* [1919], Carlo Carrà had this to say regarding the concept of reconstruction during the years between the two world wars: *Noi che ci sentiamo figli non degeneri di una grande razza di costruttori (Giotto, Paolo Uccello, Masaccio, ecc)* (We who feel ourselves to be not-unworthy sons of a great race of constructors (Giotto, Paolo Uccello, Masaccio, etc.) (quoted in [3]). On Sironi, see [42], especially chap. V.

> Sironi's efforts, more arduous and complex, have long tended to transfuse and reveal this pitiless modern beauty of the essential and geometric in paintings and in admirable drawings [that . . .] severe of line, but without schematic poverty, tend towards a new classical synthesis, not traced over the models of faded ages; definitive in spirit, but frankly ours in the forms.[17]

In certain ways the art of Sironi can be interpreted according to the 'Apollonean' lines of an unexhausted will to construct, to erect a monument that is architectonically solid, or as Sarfatti would say, concrete, simple and definitive, aimed at celebrating the classical tradition, order and man once again at the centre of the composition. On the other hand, however, the figures and objects that emerge from this mental operation, the tools of the architect and the engineer, the cubes and spheres, set squares and straightedges as well as the arches and viaducts, the objects that at first glance seem to have been created by a clear and almost optimistic will, carry with them the flavour of an unresolved break between the artist and reality, a space in which the notation of melancholy insinuates itself. Jean Clair writes:

> The merger of melancholy and geometry, of melancholic sensitivity and a *more geometrico* way of thinking goes back in this case to the neo-Platonic doctrines of the Renaissance, when the seven liberal arts were subject to the rule of the seven planets. To Saturn, as the *auctor temporum* god, the governor of time, belonged geometry, and more in general, everything that regarded the measurement of time, space and their applications. Geometry, which works with numbers, is the one who lived under the reign of Saturn ([18], p. 29).[18]

A work such as *Seated Woman in a Landscape* or *Malinconia*, painted by Sironi in 1928, becomes a modern allegory for the melancholic *umor*, described according to the classical iconography of Dürer. The artists avails himself of a classical topos, shifted through the celebrated Renaissance tradition, to render his female figure a renovated emblem of *mélania cholé*, of the incapacity to grasp reality, of modern man's sense of impotence.[19]

The art of Sironi, one of the artists who was closest to Margherita Sarfatti and was among the leading figures of the artistic renovation between the two wars, provides an opening to a broader reflection on the origins and destinies of the 1920s *retours*, also with regard to their relationship to mathematical thought. The Renaissance *modus* is recovered, we see a return to speaking about Beauty per se, of absolute Beauty, of the world of ideas, of the mathematisation of reality, and the artists answer the call. However, in Sironi's vigorous, pared down constructions, in Casorati's perspective flights and silent interiors, in Carrà's Giottesque figures and landscapes, in Severini's intellectual constructions (and we could also cite the still lives of Derain and Marussig, and many others) the formal recovery is by now without the rationalising and interpretative power that it had in the 1400s and 1500s. The architects, engineers, mathematicians, constructors and all of their

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17 MART, Archivio del '900, Fondo Margherita Sar 3.3.42. See also [43].

18 For the painting of 1928, see the intense narrative of [44].

19 See ([18], pp. 28–44). For the theme of melancholy, see the fundamental works [45–47]. For Sironi, see also [48] and [49].

progenies who populate the works of the 1920s are only apparently a triumph of order over chaos, of the renovated heroic and eternal humanism that Sarfatti spoke of: 'Modern melancholy is also a radical melancholy: it is a foreboding that no *mathesis universalis* can re-order or reunite the *disjecta membra* of reality' ([18], p. 36). At least, it seems as much to us, rereading the pages of a history that is recent and still extremely current.

(Translated from the Italian by Kim Williams)

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Mathematics, A Beautiful Elsewhere

An Unusual Art Exhibit at the Fondation *Cartier* pour l'art contemporain

Jean-Pierre Bourguignon

The exhibit took place, as one can read on the beautiful poster by artist Tadanori YOKOO, from October 21, 2011, to March 18, 2012, at the headquarters of the foundation, in the remarkable building designed by Jean NOUVEL.

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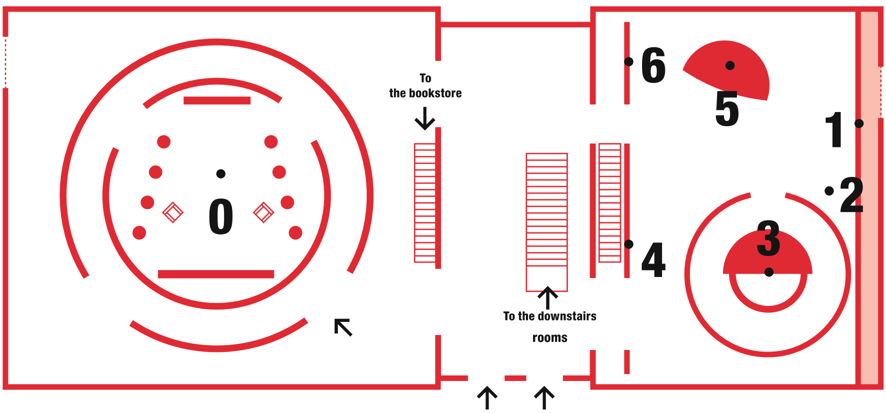
The idea of the exhibit is undoubtedly due to Hervé CHANDÈS, the Director of the Foundation who came some day in 2008 to the *Institut des Hautes Études Scientifiques* with his friend Michel CASSÉ, a well known astrophysicist also versed in poetry, to discuss with me whether I would be interested in accompanying them in the adventure of designing an exhibit about Mathematics. I said that I was ready to think about it with them, and this is why the three of use became the curators of the event.

The exhibit occupied the two floors of the building, with five principal areas. Here is the arrangement of the ground floor.



On the left hand side of the ground floor, there was the Library of Mysteries (called 0 on the map because its whole design was circular) designed by David LYNCH. The installation consisted in an an impressive structure reminding of a Greek temple. In it was shown a movie that resulted from interactions of the

artist with the mathematician Misha GROMOV presenting excerpts of reference statements by famous scientists in the course of History and its ceiling was occupied by an animation about the various scales present in the Universe. These movies were creations of David LYNCH and his team. There was also a “numerical fireplace”.

On the right hand side of the ground floor were displayed pieces due to several artists and involving different kinds of scientific activities. The overall design of this part of the room was again due to David LYNCH.

A piece was provided by the very colourful paintings of mathematical situations (4) by the artist Beatriz MILHAZES.

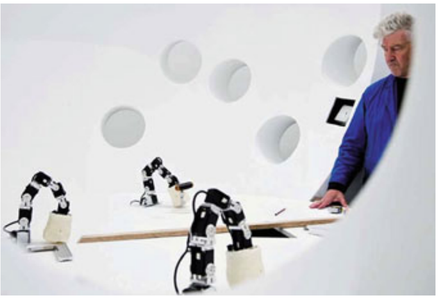
One piece of the design was a hemispherical screen (5) on which were projected mathematical phenomena, that could be appealing visually, touching different areas such as, for example, the distribution of prime numbers among natural numbers, when displayed properly.

One of the elements was, until half way through the exhibit, a window on the control room of the LEP experiment at CERN, and, for the other half, on the control room of the Planck satellite experiment bringing back images for the far away radiation in the Universe. The screen (1)[1] showed also explanations about the standard model of elementary particles, and this was a pretext taken by David LYNCH to film the hand of a physicist writing the relevant symbols, and this can be seen above.

Takeshi KITANO (6) came up with a mathematical challenge to all visitors: *"Obtain 2011 as the result of a properly formed mathematical expression involving as few as possible of the numbers 1, 2, 3, … used in this order and separated by as many mathematical symbols as one wants taken from the list shown below"*. Everybody could try on the screen and every solution found was recorded.

[1] The numbers (0–10) refer to the numbers that appear on the maps of the two floors of the exhibit.

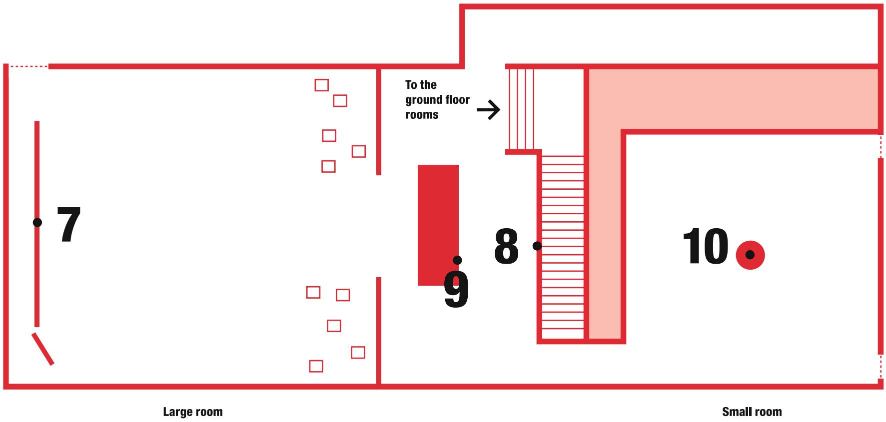
One element that attracted a lot of attention was actually a live scientific experiment proposed by Pierre-Yves OUDEYER, an engineer at INRIA in Bordeaux, who created a group of five robots (3) programmed to maximize their curiosity.



The robots can animate their bodies, listen to the sounds produced around them, both by the other robots and by the public, and produce sounds on the basis of what they identify as signs of recognition. Their face has been designed by David LYNCH, who can be seen in the picture above contemplating the robots at a time where they were not yet active. All that was happening was duly recorded and analysed to see the influence of the public on what the robots were learning. Due to the limited size of their memory, they had to be reinitialized more or less every 4 days, and this allowed to realize that their learning process did depend on the changing environment.

One more element challenging the visitors on one of the walls was to continue a quasi-periodic tiling of the plane, a concept due to Roger PENROSE.

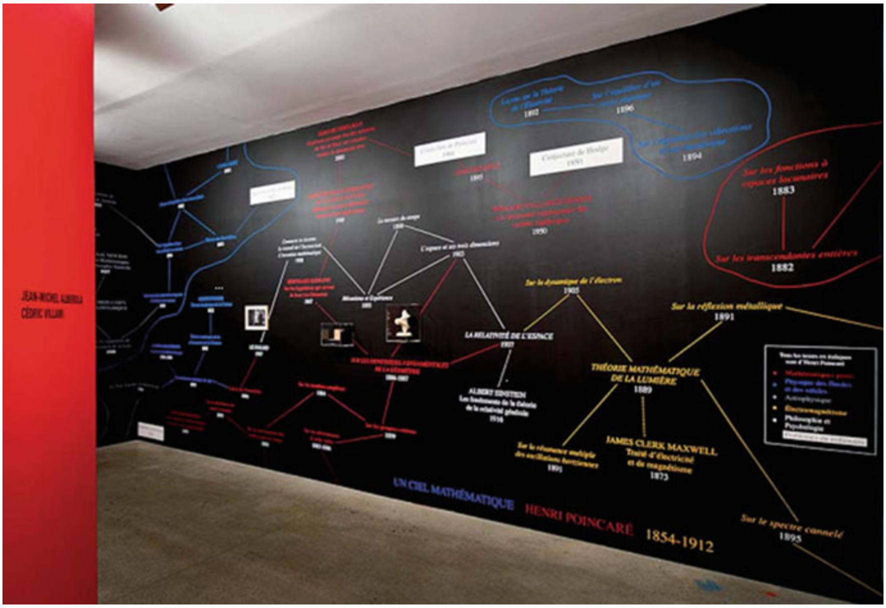
The underground floor was displaying four different pieces again due to several artists.



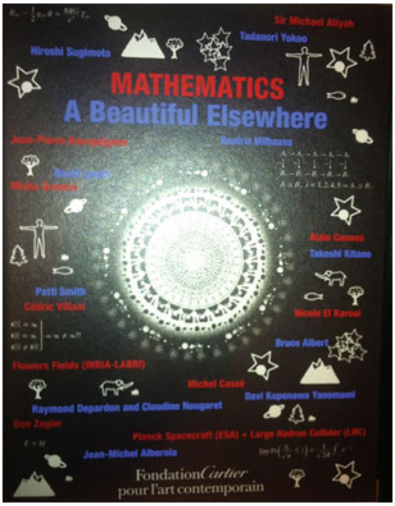
The larger room downstairs had been turned into a high quality movie theatre (7) showing on a huge screen *"Au Bonheur des maths"*, a series of nine 4-minute portraits of mathematicians by Raymond DEPARDON and Claudine NOUGARET.

The painter Jean-Michel ALBEROLA has been fascinated by the hand of mathematicians writing on blackboards. This prompted him to shoot a movie (9) of Cédric VILLANI's hand practicing this exercise familiar to mathematicians.

Jean-Michel ALBEROLA contributed another piece in the form of the *"Mathematical sky of Henri Poincaré"* (8), where stars are replaced by his main scientific works and the links represented how they related to one another and to other major pieces of science at the turn of the twentieth century.



The last room was occupied by a piece by Hiroshi SUGIMOTO, which grew from his sequence of pictures of mathematical shapes from the University of Tokyo, shown at the Fondation in 2004. This gave him the idea of realizing some of these surfaces (10) with modern engineering tools giving an even stronger appeal to them.



The catalogue of the exhibit (ISBN) contains, on top of introductory texts by the three curators, several complementary texts, often in the form of interviews of the artists or the scientists who contributed to the exhibit.

My thanks go to the personnel of the Fondation for their very professional support and to Olivier OUADAH for the pictures.

Space, Models and Geometric Fantasies

Umberto Bottazzini

1. What is space? Kant asked in the *Critique of pure reason* (1781, 2nd ed. 1787). "Space is not a conception which has been derived from outward experiences", was his answer, but it "is a necessary representation *a priori*, which serves for the foundation of all external intuitions". Thus, "Geometry is a science which determines the properties of space synthetically, and yet *a priori*", and "the principles of geometry—for example, that 'in a triangle, two sides together are greater than the third'—are never deduced from general conceptions of line and triangle, but from intuition, and this *a priori*, with apodeictic certainty" [1, pp. 39–40]. The geometry Kant was referring to was of course Euclid's geometry, as exemplified by Kant himself by quoting Euclid's first axioms repeatedly. By the end of the eighteenth century Kant's philosophy, and the relevant conception of space and geometry, became predominant in German culture.

   Although only in private communications, apparently Gauss was the first to raise doubts about Kant's ideas of space, and the *a priori*, "apodeictic certainty" of Euclidean geometry as well. Having attempted in vain to prove the truth of Euclid's postulate of parallels, Gauss confessed in a letter to Farkas Bolyai, a friend in his student days: "as for me, I have already made some progress in my work. However, the path I have chosen does not lead at all to the goal which we seek, and which you assure me you have reached. It seems rather to compel me to doubt the truth of geometry itself" [2, p. 65]. And when in 1804 Bolyai sent him a proof of the parallel postulate Gauss was quick to find a crucial error in it.

   It is difficult to know what Gauss thought of this postulate, because he limited himself to scattered remarks in letters and manuscripts. As a matter of fact, however, after 1813 the idea of a geometry different from Euclid's slowly made its way in Gauss's mind. In 1817, in a letter to the astronomer

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Wilhelm Olbers he confessed: "I am becoming more and more convinced that the necessity of our geometry cannot be proved ... Perhaps only in another life will we attain another insight into the nature of space, which is unattainable to us now". And, opposing Kant's view, he went on to say: "Until then we must not place geometry with arithmetic which is purely *a priori*, but rather in the same rank as mechanics" [3, pp. 44–45]. Thus, according to Gauss, there was no intrinsic, *a priori* necessity to Euclidean geometry: instead, it could be a system of empirically true propositions akin to mechanics. In his letter Gauss referred to a work in defense of the parallel postulate he had received from his former student Friedrich Wachter. During a visit the latter paid to Gauss in Göttingen in April 1816, they discussed the matter and on this occasion Gauss set out his ideas about what he called an "anti-Euclidean" geometry.

In the same vein, more than 10 years later, by 1829 Gauss wrote to Friedrich Bessel: "my conviction that we cannot base geometry completely *a priori* has, if anything, become even stronger". Nonetheless, he added that he would not publish anything about his ideas on space and geometry in his lifetime, fearing he would be misunderstood—being afraid, as he wrote, of "the howls of the Boeotians", i.e. the protests, if not the ridicule, which he feared he would receive from the followers of the Kantian philosophy of space. And some months later, in April 1830, Gauss repeated once more in a letter to Bessel: "it is my inner conviction that the study of space occupies a quite different place in our *a priori* knowledge than the study of quantity. Our knowledge of the doctrine of space is totally lacking in that awareness of the need for it (and its absolute truth) which is proper to the pure doctrine of quantities". Consequently, Gauss concluded, "we must humbly admit that if Number is the pure product of our mind, Space has a reality outside of our minds and we cannot completely prescribe its laws *a priori*" [3, p. 46], [4, p. 201]. viii, 201.

2. Interestingly enough, in the period of time from 1813 to 1829 attempts at establishing a non-Euclidean system of geometry came from scholars not belonging to the mathematical milieu. In 1818, through a common friend, Gauss got a short note by Ferdinand Karl Schweikart, a professor of jurisprudence at Marburg, in which there was stated that "there are two kinds of geometry—a geometry in the strict sense—the Euclidean; and an astral geometry (*astralische Grössenlehre*). Triangles in the latter have the property that the sum of their three angles is not equal to two right angles" [2, p. 76]. Under this hypothesis, Schweikart added, he could "prove rigorously that the sum of the three angles of a triangle is less than two right angles; that the sum becomes ever less, the greater the area of the triangle", and eventually, that "the altitude of an isosceles right-angled triangle continually grows, as the sides increase, but it can never become greater than a certain [constant] length" [2, p. 76]. Schweikart

concluded that Euclidean geometry holds only on the assumption that the constant is infinite. Gauss was uncommonly pleased with the note, and made known to Schweikart that he could resolve all the problems of astral geometry once the constant was known.

Following Gauss's favorable judgment, Schweikart suggested to his nephew Franz Taurinus, who was also a jurist, to pursue the researches on 'astral' geometry. However, Taurinus was deeply convinced about the necessity of the Euclidean postulate of parallels, and expressed this view to Gauss. In his reply on November 8, 1824 Gauss recognized that the theorems of the non- Euclidean geometry look somehow paradoxical. However, Gauss stated, there is nothing impossible in it, and all his efforts to find a contradiction in such a non- Euclidean geometry had been unsuccessful: "The hypothesis that the sum of three angles [of a triangle] is less than 180°—he explained—leads to a completely different geometry from ours, which is in itself consistent and which, for my part, I have developed in a thoroughly satisfying way". The fundamental question Gauss asked was not so much about the logical coherence of non-Euclidean geometry as about the 'true' nature of space and hence about the 'true' geometry corresponding to it. In fact, he went on to say: "The only thing which contradicts our reason is the fact that, if [non-Euclidean geometry] were true, there would exist in space a defined segment, although unknown to us. It seems to me, however, that, in spite of the verbal knowledge of metaphysicians, devoid of any significance, we know very little or absolutely nothing of the essence of space" [4, p. 187].

Thus, in Gauss's opinion, the possibility of conceiving an astral, anti-Euclidean or non-Euclidean geometry (to use the expressions he adopted in different periods of time) derived from the hypothesis that Euclidean geometry might not express the geometric 'truths' of physical space. Needless to say that all this seems to be in contrast with experience, and this is what it must have seemed to Gauss. But it was equally clear to him, as it was to be clear to Lobačevskij later, that our experience is local in character. Hence Gauss's attempt at finding a large scale empirical confirmation to his conjectures by measuring the angles of a large triangle like the one formed by the peaks of the mountains Brocken, Hohehagen and Inselberg—and Lobačevskij did the same thing with a triangle whose vertices were formed by celestial bodies.

In 1825 Taurinus published the *Theorie der Parallellinien* (Theory of the parallels), in which he succeeded in determining Schweikart's constant. However, sticking to his deepest convictions, he refuted the hypothesis of the acute angle (ie. non-Euclidean hyperbolic geometry)—while recognizing that contradictions were not derived from it—because "it is repugnant to the intuition" of space. One year later he published the *Geometriae prima elementa* (*First elements of geometry*) with an appendix of a spherical-logarithmic

geometry whose formulae he obtained from the usual spherical geometry by considering a sphere with imaginary radius, formulae which Gauss had already been familiar with for years. Although Taurinus limited himself to a purely formal treatment, in spite of his opinions his work contributed to feed increasing skepticism on the *a priori* need for Euclidean geometry.

By that time Gauss was working on his *Disquisitiones generales circa superficies curvas* (1827), a memoir that represents a milestone in the history of differential geometry. "This research—he wrote to the astronomer Peter Hansen in December 1825—is deeply entwined with much else, I would like to say with the metaphysics of space and I find it difficult to shake off the consequences of this" [5, 8].

In the *Disquisitiones*, Gauss introduced the idea of studying the geometry of a surface from an intrinsic point of view by considering a surface not as embedded in three dimensional space, rather "by considering surfaces from a new point of view", namely, "not as contours of bodies, but as bodies, one dimension of which is infinitely small", a sort of film "flexible but inextensible" and deformable without stretching or folding.

The introduction of curvilinear coordinates on the surface enabled Gauss to express the linear element ds through them, that is through a formula (the *first fundamental form* of the surface) which, in his words, expressed "the true starting point for the general expression of a surface". Then, having defined the "integral" curvature of a surface, Gauss proved his "theorema egregium" on the invariance of curvature by isometric transformations: "if a curved surface is developed upon any other surface, the measure of the curvature in each point remains unchanged". Thus in particular cases, "in a curved surface that is developed upon a plane, the curvature is everywhere zero". (This happens, for example, in the case of a cylinder).

The "theorema egregium" was the keystone of Gauss's program for the study of the surfaces and of the "figures constructed on them". One of the consequences of this was his "most elegant" theorem relative to an infinitesimal triangle formed by the geodesics joining three points of a surface with curvature K (variable from point to point): the total curvature of the triangle is equal to the excess or to the defect of the sum of the angles of the triangle with respect to 180°. It was at the end of this memoir that Gauss reported the results of his measurements for a triangle whose vertices were the summits of the Broken, Hohehagen and Inselberg mountains. However the empirical experience was inconclusive, for the difference of the sum of the angles from 180° "could always be considered negligible".

Did Gauss realize that there was a deep relation between non-Euclidean plane geometry and the geometry on a surface with constant (negative) curvature? Gauss renewed interest in the principles of geometry after the publication of his *Disquisitiones* seems to suggest a positive answer. Noteworthy, Gauss's

studies on the surface generated by the rotation of a tractrix on its axis—the pseudosphere, which he called "the opposite of the sphere" and which Beltrami used to build a model of Lobačevskij's geometry (see below)—date back to those years. Be as it may, the *Disquisitiones* opened a new field of research by offering the completely new approach to the study of the geometry of a surface that Riemann was able to generalize to n-dimensional manifolds.

Eventually Gauss seemed to have decided to confront "the howls of the Boeotians" by publishing what he had discovered in non-Euclidean geometry. "In the last few weeks"—Gauss wrote to Schumacher on May 17, 1831—"I have begun to put down a few of my own meditations, which are already to some extent nearly 40 years old. These I had never put in writing; so that I have been compelled three or four times to go over the whole matter afresh in my head. Also, I wished that it should not perish with me" [2, p. 67]. But in doing so he had been preceded by someone else.

3. The young János Bolyai, the son of Gauss's student fellow Farkas Bolyai, contrary to his father's advice made the theory of the parallels the preferred object of his mathematical reflections, and in 1823 he had already found some fundamental formulas of non-Euclidean geometry. By that time János wrote to his father that "[he was] determined to publish a work on parallels as soon as [he could] put it in order". His goal had not been completely attained yet but, he affirmed, "I have found things so magnificent that I was astounded. It would be an eternal pity if these things were lost [. . .] All I can say now is that I have created a new and different world out of nothing" [3, p. 52].

   Convinced by his son, Farkas exhorted him to bring to light his discoveries as soon as possible because "there is some truth in this, that certain things ripen at the same time and then appear in different places". And "since the scientific striving is only a great war, and one does not know when it will be replaced by peace one must win, if possible; for here pre-eminence comes to him who is first"[3, p. 53]. However, János's "creation" came to be published only in 1832 as an appendix to his father's volume *Tentamen juventutem studiosam in elementa matheseos* [. . .], under the long title *Appendix scientiam spatii absolute veram exhibens: a veritate aut falsitate Axiomatis XI Euclidei (a priori haud unquam decidenda) indipendentem.* The "absolutely true science of space" that Bolyai presented was a geometry which is independent of Euclid's parallel postulate, more precisely, a non-Euclidean geometry in three-dimensional space.

   The essential content of the *Appendix* can be summarized as follows: (a) the definition of parallels and their properties independent of Euclid's postulate; (b) the geometry on a sphere of infinite radius coincides with ordinary plane geometry; (c) formulae of spherical trigonometry (independent of Euclid's postulate); (d) plane trigonometry in non-Euclidean geometry, and eventually (e) squaring the circle on the hypothesis that Euclid's postulate is false.

After receiving the work of the two Bolyai, Gauss replied to Farkas on March 6, 1832: "If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it, would be to praise myself. Indeed the whole content of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last 30 to 35 years. So I remained quite stupefied" [2, p. 100].

After summarizing the reasons that for a long time prevented him from publishing his relevant results, and his eventual decision to write down all of them, Gauss went on by saying that it was a pleasant surprise for him to be spared such a trouble all the more because "it is just the son of my old friend, who takes the precedence of me in such a remarkable manner". In order to justify his claims Gauss added some theorems of his own before turning once again to the debated question of the "metaphysics of space". "It is precisely in the impossibility of deciding *a priori* between Σ and S [ie. between the non-Euclidean and the Euclidean geometry] that we find the clearest demonstration that Kant was wrong to state that space is only a form of our intuition" [4, p. 224]. Thus, according to Gauss, Bolyai's work confirmed the empirical nature of space, in other words, it provided a clear proof that "space, regardless of our intuition, must have a real meaning"—as he once wrote [6, p. 177].

Unknown to both the young Bolyai and Gauss, a system of non-Euclidean geometry had been developed by Nikolai Lobačevskij, who in 1829 published in the *Kasan Messenger* the first of a series of papers on the subject. These were followed by the essays *Imaginary geometry* (1835) and *New principles of geometry with a complete theory of the parallels* (1835–1838), by a paper on imaginary geometry published in 1837 in Crelle's *Journal* and eventually by the *Geometrische Untersuchungen zur Theorie der Parallellinien* (*Geometric researches on the theory of the parallels,* 1840), a volume which found an attentive reader in Gauss.

Lobačevskij sought to show that his non-Euclidean geometry was consistent with more general principles "obtained from nature", and "a necessary consequence of the essence of thing" which, like other physical laws could be tested "only by experience", as he wrote in his *New principles*. To this aim he took advantage of the latest astronomical observations by considering the angles of a triangle with vertices, the Earth, the Sun, and the star Sirius. But, as had Gauss in the *Disquisitiones*, he too found that the measurements were inconclusive.

Consistent with these premises, Lobačevskij affirmed that "in nature we have cognition, appropriately, only of movement, without which the sensorial sensations are impossible", and hence, geometric concepts are nothing but "artificial creations of our minds, drawn from the properties of movement; this is why space in itself, [taken] separately from us, does not exist" [7, p. 60].

Starting from the data of sensible experience, in the *New Principles* Lobačevskij assumed as primitive the concepts of body and of contact among bodies, by means of which he defined distance as the "relative position" of two points and, consequently, the notions of sphere, plane, and line. Like Bolyai had done, Lobačevskij proceeded to develop "absolute" geometry, independent of the parallel postulate. Then he stated the fundamental proposition: "Lines that emanate from a point C, either intersect a given line in the same plane, or never meet it, no matter how much they are prolonged" [7, p. 92]. Among these are the parallels to the given line, which divide the plane into four parts: into two opposite parts with respect to C would be included the divergent lines, and in the two others, convergent lines.

A decisive role in Lobačevskij's theory is played by the *angle of parallelism* $\Pi(p)$. This is defined as the inclination of a line with respect to the perpendicular p to another line, parallel to the first one. If $\Pi(p)$ is a right angle, one has ordinary, Euclidean geometry; if instead it is less than a right angle, one has "imaginary" geometry. More precisely, Lobačevskij stated, if $\Pi(p) < \pi/2$, "we can represent a curve, which we will call a *horocycle"* (limiting curve) which can be thought of as a curve orthogonal to a bundle of parallel lines (the tangent to it in a point P is orthogonal to the line of the bundle passing through P). In ordinary geometry such a curve is a line, while in "imaginary" geometry a horocycle can be thought of as the limit of a circle when its center is extended to infinity along the direction of the lines of the bundle. Following this, Lobačevskij was able to introduce in a purely geometrical way a unit of absolute measure of segments, which can be defined as the distance x between two horocycles, relative to the same bundle of parallels. It was this singular property of the 'Lobačevskij plane' that Gauss was thinking of, when he wrote to Taurinus that astral geometry implied the existence in space of a segment defined "even if it is not known to us". In a manner analogous to his construction of the horocycle, Lobačevskij introduced the concept of *horosphere* "as the limit to which the sphere tends when its radius is increased, while the maximum circles on it are transformed into horocycles" (a horocycle lying on a horosphere divides it into two parts).

Although supported by Gauss's open recognition, Lobačevskij's theories gained poor attention by the mathematicians. On the other hand, if, as it seemed to have been experimentally confirmed, the 'true' geometry of space was Euclidean, what was the fate of Bolyai's and Lobačevskij's theories? The horocycles degenerate into a line, the horospheres into a plane, and that entire bold theoretical construction collapses on its foundations. This conviction enjoyed a long life among mathematicians: rather than announce a "new world", as the young Bolyai had hoped, their work ended up very quickly being somewhat forgotten, placed in the category of the bizarre and extravagant outcomes that from time to time are occur at the margins of scientific research.

4. Toward the middle of the century, the problem of space and its mathematical representation was confronted on a completely different basis by Bernhard Riemann. Already in his 1851 Dissertation on the foundations of the theory of complex functions he introduced the idea of *Riemann surface* that defies geometric intuition. He claimed that "it will be easy to speak of superimposed surfaces" in such a way that "the place of a point O extends many times over the same part of the plane" by assuming, however, that the multi-sheeted parts of the surface are not connected along a line, so that a folding of the surface or a division in superimposed parts cannot occur [8, p. 219]. For such surfaces Riemann introduced topological concepts like the order of connectivity:

Model of a threefold connected surface (Modellsammlung, Mathematisches Institut - Universität Göttingen)

Commenting on this, Ahlfors stated that the idea of Riemann surface "is a skillful fusion of two distinct and equally important ideas: (1) a purely topological notion of covering surface" and (2) "an abstract conception of the space of the variable with a local structure defined by an uniformizing parameter" [8, p. 219] that comes to the foreground when dealing with branch-points (see figures below). While refusing to take the n-sheeted Riemann surface as the proper foundation of function theory, because "it would be difficult to deal with them when there are infinitely many sheets", Weierstrass recognized that Riemann "had a mathematical imagination like no-one else he [Weierstrass] had ever known" [9, p. 474]. Riemann himself provided a further essay of his geometrical imagination with his 1854 inaugural lecture on *Über die Hypothesen welche der Geometrie zu Grunde liegen* (On the hypotheses that lie at the foundations of geometry) devoted to n-dimensional manifolds.

Model of a simple branch-point (Modellsammlung, Mathematisches Institut - Universität Göttingen)

Model of a branch-point of 2nd order (Modellsammlung, Mathematisches Institut - Universität Göttingen)

The idea of a n-dimensional space had already begun to emerge in mathematics at that time, and apparently Gauss himself was familiar with such an idea. Indeed, the 'metaphysics' of multi-dimensional spaces seems to occur continuously in his reflections in the last part of his life. According to his colleague Sartorius von Walterhausen, Gauss once said that "we may imagine a sort of being aware of only two dimensions: perhaps those above us may look down in a similar fashion to us" [10, pp. 117–118] thus offering evocative images that remind us of Abbott's *Flatlandia*. There is hardly any doubt that it was Riemann who reaped Gauss's theoretical heritage in his inaugural lecture.

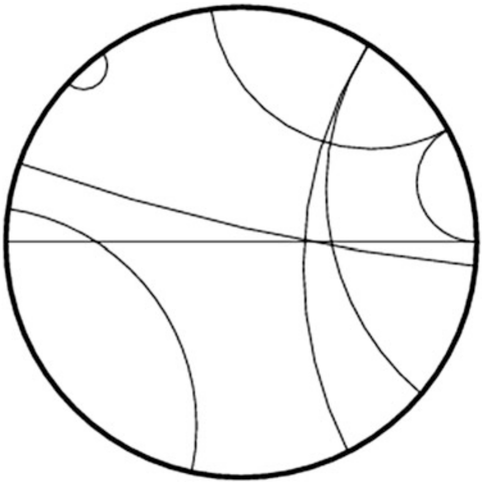
There Riemann began by stating that a n-dimensional manifold is susceptible of different metric relations so that ordinary space is only a special case of 3-manifold. Accordingly, Euclidean geometry was not a “necessity” but merely an empirical “certainty” based on “hypotheses”, no matter how high their probability within the limit of observation. By generalizing Gauss’s intrinsic geometry of surfaces to n-dimensional manifolds Riemann defined a metric on a manifold under the hypothesis that the length of a line is independent of its position on the manifold. Accordingly, the (square of the) linear element ds^2 could be given by the positive, homogeneous quadratic form $\sum g_{ij}\, dx_i\, dx_j$ where the coefficients g_{ij} are the continuous functions of $x_1, \ldots, x_n$ such that $g_{ij} = g_{ji}$.

Given this, he was able to extend Gauss’s concept of curvature to the manifolds. The entire treatment of Riemann had a “local” character, that is, it was essentially concerned with the study of geometry in the immediate neighborhood of a point P of the manifold. In the concluding part of his lecture Riemann pointed out that there is “an essential difference between metric and extensive relations”, which becomes decisive when “the constructions of space are extended to the incommensurably large”: in such a case, it implies “the distinction between the unbounded and the infinite: the first belongs to the extensive relations, the second to the metric relations”. That space would be unbounded was for Riemann, “an empirical certainty much greater than any other empirical fact.” But given that, he continued, “it does not follow that space would also be infinite”. Instead, assuming that bodies are independent from their positions in space, space “would be necessarily finite, as soon as its curvature had a positive value”. (This allowed to consider non-Euclidean elliptic geometry).

5. The publication of Riemann’s lecture in 1867 convinced Beltrami to publish his own paper *Saggio di interpretazione della geometria non euclidea (Essay on an interpretation of non-Euclidean geometry,* 1868) where he provided an Euclidean model for Lobačevskij’s plane geometry. This was given by a surface of constant, negative curvature, the pseudosphere generated by the rotation of a tractrix around its axis. He himself provided a paper model of (a piece) of such a surface, and sent it to his friend Cremona—and currently kept in the library of the Department of mathematics, University of Pavia (see figures below).

Model of a pseudosphere (Modellsammlung, Mathematisches Institut - Universität Göttingen)

Beltrami’s own paper model (with the drawings of angle of parallelism and horocycle)



Poincaré's disk

Beltrami's model had provided a proof of the (relative) consistency of non-Euclidean (Lobačevskij's) geometry. Some 10 years later it was also to inspire Poincaré who, in a flash of geometrical intuition, realized that it offered the appropriate geometry in the complex upper half plane (or, equivalently, a disk) for his Fuchsian functions.

In Poincaré's disk the space is given by the interior of the disk, 'straight lines' are arcs of circumferences orthogonal to the limit circle. (Needless to remind that this model has found a number of variations in Escher's hands). Echoing Gauss's fantasy without knowing it, Poincaré imagined that, under appropriate, non-contradictory physical conditions, in such a 2-dimensional world 2-dimensional inhabitants could live in a non-Euclidean geometric setting.

From a mathematical point of view a natural question arises: could Poincaré's disk be isometrically embedded in space? The answer is negative: as Hilbert proved in 1900, there is no such isometric embedding (of class C^2). In other word, it turns out that Beltrami's model is local, in the sense that the whole Lobačevskij plane cannot be in general represented by an analytical surface with constant negative curvature.

According to Poincaré, the object of a particular geometry reduced to nothing else than the study of a particular group of transformations. Unknown to him, this approach had been proposed by Felix Klein in his *Erlangen Program* (1872) where he stated that to study the geometry on a manifold was to study the properties left invariant by the action of a group of transformations. This allowed him to consider non-Euclidean geometries from a unitary point of view. Indeed, in 1871 in a paper with the telling title *Über die sogenannte Nicht-Euclidische Geometrie* (*On the so-called non-Euclidean geometry*) he had

proven that both Euclidean and non-Euclidean geometry could be considered as special cases of projective geometry depending on the choice of a fundamental surface of the second order (the Absolute). Namely, three cases occur: (1) if the fundamental surface is imaginary one has elliptic geometry; (2) if the fundamental surface is real (and not ruled) one has hyperbolic geometry; eventually, (3) if the fundamental surface degenerates into a imaginary curve one has parabolic geometry (ie. ordinary Euclidean geometry). As an immediate consequence, this provided yet another proof that non-Euclidean geometry is consistent if and only if Euclidean geometry is consistent thus putting an end to a controversial, long debated matter.

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Visible Harmonies: Mathematical Models

Michele Emmer

From Plato to Kepler

The hall of the 1986 Venice Biennale dedicated to *Art and Science* opened with works by Lucio Saffaro (the artist whose works are featured on all the covers of books in the *Imagine Math* book series), along with Felice Ragazzo's reconstruction of Kepler's model of how the universe works, when Kepler still believed that the orbits of the planets were circular and circumscribed by the regular solids first described by Plato [27, 5, 28, 6, 8, 9].

The fascination of the Platonic solids began thousands of years ago and still continues today. It is a fascination tied to the search for the mathematical models that reflect the universe and the forces that compose it. Such models are images of perfection and harmony, based as they are on mathematical structures. As Plato said in his dialogue *Timaeus*: 'We shall not be willing to allow that there are any distinct kinds of visible bodies fairer than these'. This is how Plato describes the five regular solids:

> ... when the world began to get into order, fire and water and earth and air had only certain faint traces of themselves, and were altogether such as everything might be expected to be in the absence of God ... God made them as far as possible the fairest and best, out of things which were not fair and good. ... [A]s is evident to all, fire and earth and water and air are bodies. ... [N]ext we have to determine what are the four most beautiful bodies which are unlike one another, and of which some are capable of resolution into one another; for having discovered thus much, we shall know the true origin of earth and fire and of the proportionate and intermediate elements. And then we shall not be willing to allow that there are any distinct kinds of visible bodies fairer than these. Wherefore we must endeavour to construct the four forms of bodies which excel in beauty, and then we shall be able to say that we have sufficiently apprehended their nature [32, §53b-e].

Here then are the five mathematical models that are the models of nature:

> [F]our equilateral triangles, if put together, make out of every three plane angles one solid angle, being that which is nearest to the most obtuse of plane angles. And out of the

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> combination of these four angles arises the first solid form which distributes into equal and similar parts the whole circle in which it is inscribed [32, §54e-55a].

This is the tetrahedron, which is the element of fire.

> The second species of solid is formed out of the same triangles, which unite as eight equilateral triangles [32, §55a].

This is the octahedron, the element of air.

> The third body is made up of one hundred twenty triangular elements, forming twelve solid angles, each of them included in five plane equilateral triangles, having altogether twenty bases, each of which is an equilateral triangle [32, §55a-b].

This is the icosahedron, the element of water.

> But the isosceles triangle produced the fourth elementary figure, which is compounded of four such triangles, joining their right angles in a centre, and forming one equilateral quadrangle ... the figure of the body thus composed is a cube, having six plane quadrangular equilateral bases [32, §55b-c].

This is the cube, the element of earth (Fig. 1).

> There was yet a fifth combination which God used in the delineation of the universe [32, §55c].

This is the final solid, the dodecahedron. This last solid plays a special role. Historian Carl Boyer hypothesized that it was the Pythagorean cult of the geometric

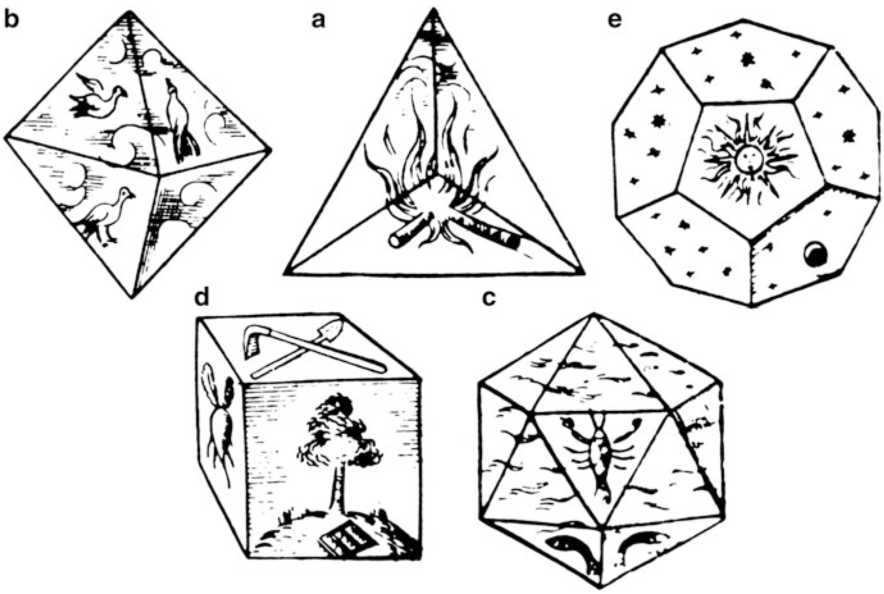


Fig. 1 Johannes Kepler, *Harmonices Mundi*, 1619

shape of the dodecahedron that led Plato to adopt this regular solid as the symbol of the universe [4].

From the time Plato wrote this words about the solids, and Euclid formalised their properties in *The Elements*, the influence of the regular solids on the culture of what would become Europe is beyond doubt. The culminating moment was the Renaissance.

The mathematician Luca Pacioli, who was perhaps a student of Piero della Francesca, incorporated Piero's treatise on the regular solids in his famous book of 1509, *De divina proportione*. That book owes much of its fame to the fact that the 60 plates of the regular, semi-regular and stellated solids were '*facte e formate per quella ineffabile mano sinistra a tutte discipline mathematici accomodatissima del prencipe oggi fra i mortali, pro prima fiorentino, Leonardo da Vinci*' (made and formed by that ineffable left hand so very well suited to all mathematical disciplines belonging to the leading figure of all mortals today but above all Florentine, Leonardo da Vinci). Piero della Francesca painted a portrait of his student Luca Pacioli, depicting him in the guise of St. Peter in his Brera Madonna (so-called because it is housed today in the Pinacoteca di Brera in Milan). There is also another portrait completed in the years 1498–1500, attributed to Jacopo de' Barbari (the same artist mentioned often in this present volume, creator of the famous plan of Venice of 1500) today housed in the Capodimonte Museum in Naples. This second painting is particularly interesting because it shows, in the upper left-hand corner, a depiction of a model, perhaps in glass, of a semi-regular solid: the rhombicuboctahedron which corresponds to plate XXXV in *De divina proportione*, entitled *Vintisex basium planus solidus*. The painting also features another solid in the lower right-hand corner: a dodecahedron, called a *Duodecedron planus solidus* in Pacioli's book. Pacioli's left hand is pointing to a page in Euclid's *Elements*. The interested reader will find a possible interpretation of the painting in the article by Enrico Gamba [20].

An important role is played in treatises on perspective not only by the regular solids, but by other kinds of solids as well, from the semi-regular or Archimedean solids and stellated solids, to mazzocchi and other curious polyhedral shapes [7, 13]. Such treatises began to multiply at the beginning of the Quattrocento following the studies and experiments conducted by the Florentine architect Filippo Brunelleschi, which we know thanks to the first treatment of the subject, written by Leon Battista Alberti in his *De Pictura* (1434–1436). Knowing how to draw three-dimensional solids in perspective was essential for those who aspired to master perspective. Geometric models were thus an essential component in the renewed study of geometry during the Renaissance.

We need cite only two of the most famous examples, the treatise by Wenzel Jamnitzer entitled *Perspectiva corporum regularium*, printed in Nuremberg in 1568, and that by Lorenzo Sirigatti entitled *La pratica della prospettiva*, printed in Venice by Girolamo Franceschi in 1596 (Figs. 2 and 3).

The Museo Galileo of the Institute and Museum of the History of Science in Florence put together an interesting website entitled "Bibliotheca Perspectivae" about the treatises of perspective, featuring a collection of digital versions of original

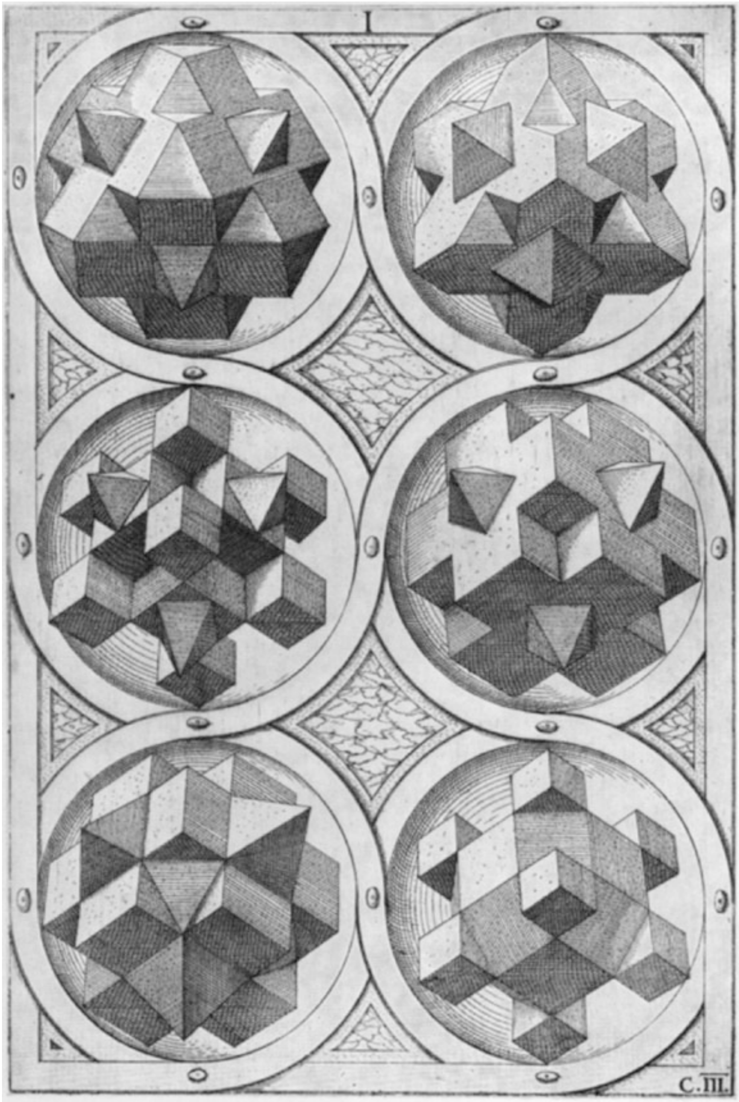


Fig. 2 Ridolfo Sirigatti, *La pratica della prospettiva*, 1596

works, perspective images from the Renaissance period, and more [1]. They write, ‘Analogous to what has been done or is being done for treatises on architecture, writings on art and sources for the history of science, the aim is that of providing scholars with a library that first of all collects the written sources of the period comprised between the fifteenth and the eighteenth centuries’. A part of that website is dedicated to regular and irregular geometric bodies.

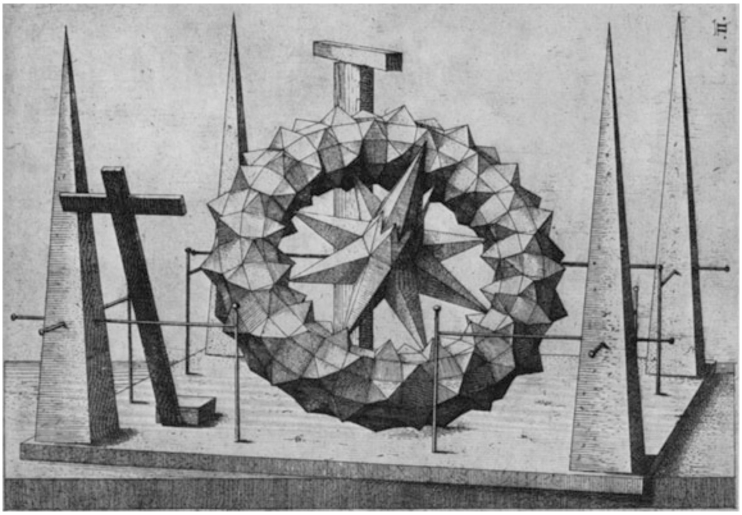


Fig. 3 Wenzel Jamnitzer, *Perspectiva corporum regularium*, 1568

The treatises of Piero della Francesca, Jamnitzer and Sirigatti, referred to by Saffaro in his earliest works regarding mathematics and art, inspired him to create a long series of paintings of polyhedra. Saffaro is a leading figure in an interesting discovery of the history of polyhedra.

It is believed that Kepler was the first to note that the regular solids form duals of each other. In his 1619 treatise entitled *Harmonices mundi*, Kepler described a solid that he calls *stellarum duodecim planarum pentagonicarum* like this:

> *Habet hoc conjiugium et stellam solidam, cujus genesis est ex continuatione quinorum planorum dodecaedri, ad concursum omnium in puncto unico.*
>
> (This marriage also has a solid star, the generation of which is from the continuation of the five faces of the dodecahedron so that they all meet at a single point.[1])

The solid that he is speaking of is a star dodecahedron, and Kepler is credited with having discovered it. It is called a 'star dodecahedron', or 'stellated' because a regular pyramid is constructed on each of the dodecahedron's face. Still in 1619 Kepler published the first perspective representation of two regular star dodecahedra, but one of the bodies obtained by Kepler was also depicted, this time in mosaic, on the pavement of St. Mark's Basilica in Venice. This is attributed to Paolo Uccello, who is believed to have made it while he was in Venice in the years 1425–1430, that is, quite a long time before its official mathematical discovery. This is quite a rare phenomenon, and underlines the fact that the artists of the day can be considered the true mathematicians of that epoch (Fig. 4).

---

[1] Eng. trans, Johannes Kepler, *The Harmony of the World*, E. J. Aiton, A. M. Duncan and J. V. Field, Philadelphia, American Philosophical Society, 1997, p. 397-398.

Fig. 4 Paolo Uccello, *Dodecaedro stellato,* 1425–1430

Lucio Saffaro became aware of the stellated polyhedron attributed to Paolo Ucello in 1970 [34, 35], and it seemed incredible to him that no mathematician had ever considered it before. Later he discovered that this particular polyhedron was mentioned with evident amazement in a work published in 1876 by the German historian Siegmund Günther [21]. Saffaro noted that the slight defects present in the depiction of the polyhedron can be ascribed to an attempt to lend greater spatial depth to the image, and are easily rectified. It can also be presumed that the corrections were made by the mosaic craftsmen during execution unbeknownst to Paolo Uccello, who supposedly only provided them with a geometric model of the polyhedron. Uccello's image of the star dodecahedron became famous in 1986 because it was chosen as the symbol of that year's Venice Biennale, which was dedicated to the theme of Art and Science, and appeared on the poster and the cover of the catalogue (see [26, p. 24]). However, that has done nothing to stop the progressive disintegration of this work of art, which lies in the pavement of one of the entrances to the Basilica, where it is walked on by thousands of visitors each day.

Saffaro later noticed that in the pavement of a chapel of the Church of St. Pantalon, also in Venice, there are two identical marble intarsias that represent the second star dodecahedron of Kepler. Their author remains unknown today, but it could again be Paolo Uccello.

In his *Harmonices mundi* Kepler dealt with polyhedron in depth, proving among other things that there are 13 semi-regular solids, and constructing, in addition to the two stellated polyhedra, the famous *stella octangula* which, as the mathematician Coxeter noted, had already made an appearance in the *De divina proportione*, where Luca Pacioli calls it *octaedron elevatum*. The *stella octangula* is obtained from the combination of two tetrahedra (the octahedron is self-dual, that is, it is a dual

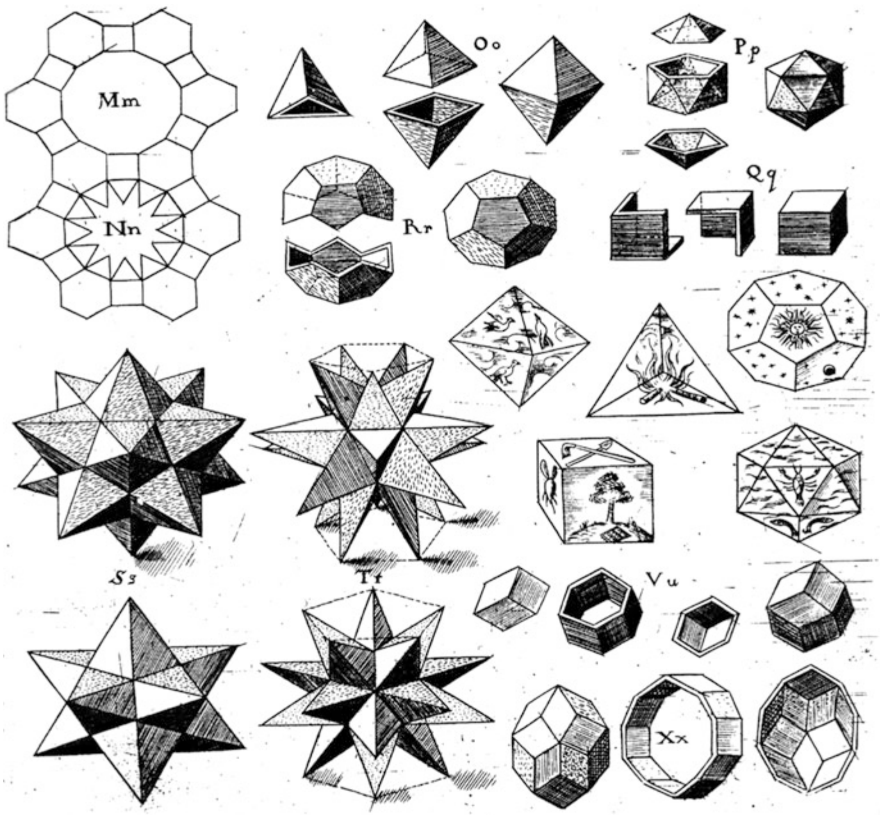


Fig. 5 Johannes Kepler, *Harmonices Mundi*, 1619

of itself: for this reason is was called celibate, or hermaphrodite, by Kepler, who attributed a gender to each polyhedra). Kepler described it thus:

> XXVI Propositio: Addi possunt congruentijis perfectissimis regularibus, duae etiam aliae congruentiae, stellarum duodecim planarum pentagonicarum: et duae semisolidae, stellarum octangulae, et decangulae.
>
> (Proposition XXVI: We may add to the most perfect regular congruences two further ones, each involving twelve star pentagons, and two semisolid congruences, of star octagons and star decagons.[2]) (Fig. 5)

Kepler was very interested in polyhedra because he was seeking possible ties with astronomy. In the preface to his *Mysterium cosmographicum* of 1596, he wrote:

[2] Eng. trans, Johannes Kepler, *The Harmony of the World*, E. J. Aiton, A. M. Duncan and J. V. Field, Philadelphia, American Philosophical Society, 1997, p. 116.

> In this little book, dear reader, I set out to prove that the Most Excellent Creator, in the creation of this our mobile world and in the arrangement of the heavens, looked to those five regular solids that have enjoyed such great fame from the times of Pythagoras and Plato up to our day, and that according to their nature he formed the number and proportion of the heavens and the ratios of celestial motions.

New Models

Over the course of centuries the panorama of shapes and mathematical models of surfaces increasingly broadened to the point where mathematicians believed that it was opportune to construct models of the new surfaces that had been obtained. I do not intend to go into detail about the history of geometric models that were constructed, above all in Germany, beginning in the second half of the 1800s. For a more detailed investigation, I refer to the readers, for example, to the article by Nicla and Franco Palladino, who write:

> The mathematical models realised (essentially in Europe) in an interval of time that can be defined, with a good approximation, as the period between the beginning of the second half of the 1800s and the 1930s, were the product of a cultural undertaking that involved some of the most active mathematics institutes present in European universities and polytechnic schools. This undertaking saw the participation of first-rate figures working in the mathematical sciences and was made fertile by interactions with the research and didactics at the university level (with favourable consequences for pre-university teaching). The undertaking also involved the most important museum centres of Europe; in addition, the mathematical models, as they spread, even forged interesting links with the sector of figurative arts and the world of cinema [see below on the films by Sinisgalli] when they fell into the sensitive gaze of sculptors, painters and set designers [30].

It is of the relationships between mathematical models and some of the artists of the twentieth century that I intend to speak. Let us begin with Henry Moore, the famous English sculptor [31].

> The great French mathematician Gaspard Monge (1746–1818) invented what can be called descriptive geometry and illustrated his discoveries using surfaces created by stretching strings across a curved frame [41].
>
> Developments in the organisation of mathematics are the key to the actual production of models. In France, the founding of the École Polytechnique was crucial ... The person who created descriptive geometry appears to have made the first surface models ... Monge introduced geometrical thinking back into mathematics, putting it on an equal basis with analysis and inspiring the revival of pure geometry and illustrated his discoveries using surfaces created by stretching strings across a curved frame. (In mathematics these are called ruled surfaces because through every point there is at least one, sometimes more than one, straight line which lies on the surface.) We know he had two models of silk thread which were extant in 1814, however, unlike the extant models they were fixed [42].
>
> The pedagogical application of Monge's string models was taken up by his pupil Thèodore Olivier (1793-1855) who used articulated frames with moveable components, to visualise a wide variety of surfaces simultaneously using different coloured strings [41].

Fig. 6 Fabre de Lagrange, *String Surface Model: Hyperboloid and Asymptotic Cone*, 1872. Courtesy of Science Museum/Science & Society Picture Library, London

Théodore Olivier (1793–1853), a professor of descriptive geometry at the Conservatoire National des Arts et Métiers in Paris, designed a series of string models of ruled surfaces that overcame the difficulty of the rigidity of Monge's models: 'These models could be distorted and rotated thus providing a variety of geometrical configurations' [41]. Olivier's personal collection of models was sold in 1853 to Union College Schenectady in the state of New York.

Olivier's activities were carried on by Fabre de Lagrange [30, pp. 61–62]. The Science and Art Department of the Committee of Council on Education, headquartered at the South Kensington Museum—today the Science Museum—in London, decided on 22 January 1875 to create a Loan Collection of Scientific Apparatus 'to include not only apparatus for teaching and for investigation, but also such as possessed historic interest on account of the persons by whom, or the research in which, it had been employed'.

In 1876 there was an exhibition which included the publication of a catalogue in which appears a 'Collection of Models of Ruled Surfaces, constructed by Fabre De Lagrange, in 1872, for the South Kensington Museum'. This comprised 45 ruled surfaces, created in string, for which a special catalogue was printed, today available on the Internet [18].

The Science Museum in London has 30 of the models by Fabre de Lagrange. An interesting aspect of why in that period models of surfaces had begun to be constructed is explained by Jane Wess, who works at the museum: (Fig. 6)

> The makers of the mathematical surface models were inspired by the usefulness of mathematics as a foundation for the industrial era . . . The sculptures are constrained by the aesthetic sensibilities of the sculptor, but the mathematical surface models are constrained by mathematical forms described by equations. However, interest in surfaces and the

> construction of surface models only began at the end of the eighteenth century, and went hand in hand with a change of social use and therefore audience [42].

In the catalogue of the collection of Fabre de Lagrange's models in the Science Museum, C. W. Merrifield writes:

> These surfaces, on account of the facility with which they can be constructed and represented, and of the ease with which their intersections can be determined, are of more consequences than any others in the geometry of the Industrial Arts [18].

The words by Jane Wess quoted above are not taken from the catalogue of an exhibition of mathematical models, or better, it is true that there were models by Fabre de Lagrange on display, but it was essentially an art exhibition. Because mathematics and art have a long historical relationship, the two disciplines have often been entwined.

A Brief History

In the second half of the nineteenth century geometry had mutated significantly. In a letter of December 1799 Carl Friedrich Gauss (1777–1855) wrote to Farkas Bolyai (1775–1856) about his attempt to prove the fifth postulate of Euclid's *Elements*, starting from a *reductio ad absurdum*: "My works are very advanced but the way in which I am moving is not conducive to the aim I am looking for, and that you say to have reached. It rather seems to put in doubt the exactness of geometry." Gauss's results on this particular topic were never published in his lifetime. In 1827 he published the *Disquisitiones generales circa superficies curvas*, in which he introduced the idea of studying the geometry of a surface in a *local way* without considering its immersion in a three-dimensional space, studying the invariant properties of the surfaces. He also introduced the idea of the curvature of the surface.

Between 1830 and 1850 Nikolai Lobachevsky (1792–1856) and Bolyai built the first examples of non-Euclidean geometry, in which the famous fifth postulate by Euclid was not valid. Not without doubt and conflicts, Lobachevsky would later call his geometry (which today is called non-Euclidean hyperbolic geometry) 'imaginary geometry', because it was in such strong contrast with common sense. For some years non-Euclidean geometry remained marginal to the field, a sort of unusual and curious genre, until it was incorporated into and became an integral part of mathematics through the general ideas of G. F. B. Riemann (1826–1866). In 1854 Riemann gave his famous lecture entitled *Ueber die Hypothesen welche der Geometrie zür Grunde liegen* (On the hypotheses which lie at the foundation of geometry) before the faculty of the University of Göttingen (it was not published until 1867). In his presentation Riemann described a global vision of geometry as the study of varieties of any dimension in any kind of space. According to Riemann, geometry didn't necessarily have to deal with points or space in the traditional sense, but with sets of ordered *n*-ples.

In 1872 in his inaugural speech after becoming professor at Erlangen (known as the *Erlangen Program*), Felix Klein (1849–1925) described geometry as the study of the properties of figures that were invariant with respect to a particular group of transformations. Consequently each classification of the groups of transformations became a codification of the different types of geometry. For example, Euclidean plane geometry is the study of the properties of the figures that remain invariant with respect to the group of rigid transformations of the plane, a group comprised of translations and rotations.

Jules Henri Poincaré maintained that:

> *The geometrical axioms are ... neither synthetic a priori intuitions nor experimental facts*. They are conventions. Our choice among all possible conventions is *guided* by experimental facts; but it remains *free*, and is only limited by the necessity of avoiding every contradiction, and thus it is that postulates may remain rigorously true even when the experimental laws, which have determined their adoption, are only approximate. In other words, *the axioms of geometry* (I do not speak of those of arithmetic) *are only definitions in disguise*. What, then, are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true and if the old weights and measures are false, if Cartesian coordinates are true and polar coordinates are false. One geometry cannot be more true than another; it can only be more convenient. Euclidean geometry is and will remain the most convenient [33], pp. 58-59.

Poincaré, in *Analysis Situs*, published in 1895, is also responsible for the official birth of the sector of mathematics that today is called *topology*.

It is easy to see how mathematics has contributed to changing our concept of space—the space in which we live and the idea of space itself, because mathematics is not merely a means of measurement and recipes, but has contributed, if not determined, the way in which we understand space on earth and in the universe.

It is important to mention that the discovery (or invention) of non-Euclidean geometry and of the higher dimensions (from the fourth on)—in short, the new idea of space—is one of the most interesting examples of the profound repercussions that mathematical ideas can have on humanistic culture and on art [12, 14, 16, 10, 11]:

> During this period mathematicians began to produce less regular and more startling geometric figures. Many of these new and exciting mathematical ideas filtered into the public sphere and sparked the imagination of writers and artists.

Linda D. Henderson has analysed in detail the influences that new mathematical ideas had on art at the beginning of the twentieth century. She published a first book in 1983, entitled *Non Euclidean Geometry, the Fourth Dimension and Modern Art* [22], and in 2013 finally published a second edition with a new 96-page introduction [24]. Henderson also published the book of reference on Marcel Duchamp, in which she also examines the artist's interest in science, especially in mathematics [23]. In particular, she leaves no doubt that mathematical models influenced the surrealists and constructivists.

Returning to the exhibition I was speaking about, Barry Phipps of the University of Cambridge, the curator of the exhibition of Henry Moore, writes:

> The influences of these encounters between avant-garde art and mathematical extends to Britain through a number of routes ... Henry Moore was using plaster and string to create

strange and beautiful forms. Indeed, Moore stated on several occasions that the use of string in his sculpture, which began in 1937, was influenced by seeing models at the Science Museum in London (Fig. 7):

> 'I was fascinated by the mathematical models I saw there, which had been made to illustrate the difference of the form that is halfway between a square and a circle. One model had a square at one end with 20 holes along each side . . . Through these holes rings were threaded and lead to a circle with the same number of holes at the other end. A plane interposed through the middle shows the form that is halfway between a square and a circle . . . It wasn't the scientific study of these models but the ability to look through the strings as with a bird cage and see one form within the other which excited me' (quoted in [32], pp. 5-6).

The exhibition 'Intersections: Henry Moore and stringed surfaces' was mounted by the Royal Society of London and ran from April to June 2012, was curated by Barry Phipps, and was organised in collaboration with the Henry Moore Foundation, the Science Museum, and the Isaac Newton Institute for Mathematical Sciences. On display were about ten of Fabre de Lagrange's models of ruled surfaces, some of the pages of sketches by Moore done when he went to see the mathematical models, and about ten small sculptures made by Moore between 1937 and 1939 (Fig. 8).

Moore was not the only one interested in mathematical models. In those same years the artist Barbara Hepworth had an opportunity to see mathematical models at Oxford. Some of Hepworth's work from those years was directly inspired by mathematics, as shown both by the shapes and the use of string.

These are the years in which constructivists such as Naum Gabo and his brother Antoine Pevsner were also very interested in mathematical models.

It should be added that after the great period of activity dedicated to mathematical models in France, in Germany as well, in the second half of the nineteenth century, there was a great interest in building models of the new surfaces that had been discovered in the field of research. In particular, in 1868 Julius Plücker created a vast collection of models. Nicla and Franco Palladino write:

> A few years after the unification of Germany (which occurred in 1871 with the formation of the *Deutsches Reich*) the substantial initiative to construct mathematical models assumed great scientific, institutional and didactic importance, achieving its own well-defined autonomy. The production of mathematics, physics, technical-mechanical and geodesic institutions in German universities and polytechnic schools was organised and further motivated, and thus, a company (a publishing house, that of Ludwig Brill, purposely founded in Darmstadt in 1877 at the behest of Felix Klein and Alexander Brill and then continued thanks to Martin Schilling) began to act as a collecting centre with a single catalogue subdivided by series almost always comprised of models that could be traced back to the same scientific theme, conceived and often constructed by a given institute under the guidance of a given professor. The various editions of the catalogue, which was not presented as a simple list of pieces, but was enriched by detailed information about the argument involved and references to the scientific essays that inspired them, represented the descriptive *summa* of the *system of plastic models* that flourished in Germany for some forty years . . . The name of the catalogue was *Katalog mathematischer Modelle für den höheren mathematischen Unterricht* (Catalogue of Mathematical Models for Higher Mathematics Teaching). It was divided into two parts. In the first the models were ordered by series; in the second they were grouped homogeneously according to their scientific relations. The models (especially those in plaster, which were certainly superior, not only because of the

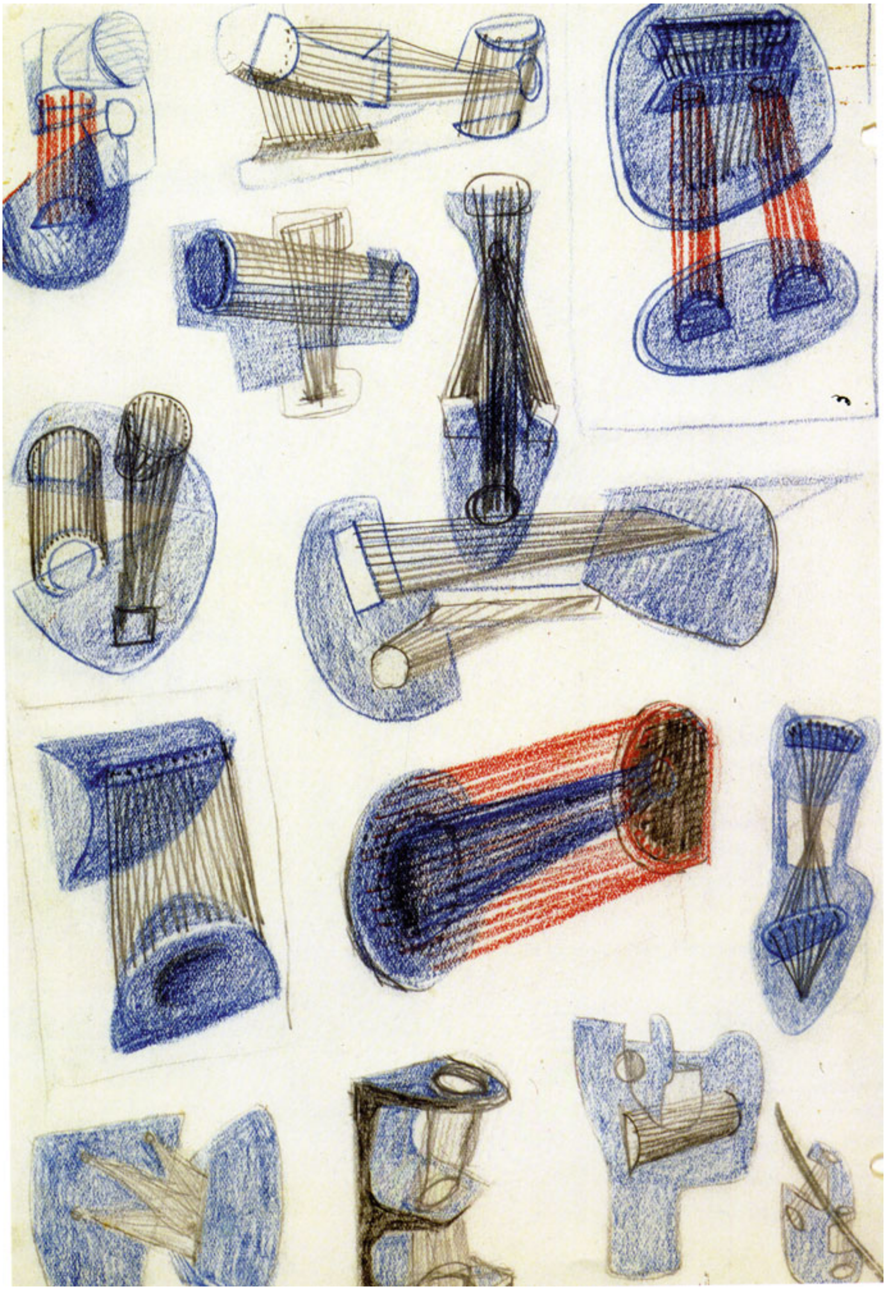


Fig. 7 Henry Moore, *Ideas for Stringed Figure Sculptures*, 1937. *Pencil, grayon, pen and ink on cream medium-weight wove*. Reproduced by permission of *The Henry Moore Foundation*

Fig. 8 Henry Moore, *Mother and Child*, 1938. *Lead and yellow string*. Reproduced by permission of *The Henry Moore Foundation*

mathematical theory that they interpreted but also because of the quality of the material and the craftsmanship) were distributed in Germany and later throughout Europe (to various extents from Spain and Portugal to Russia, from Sweden to Italy), pushing other earlier, different realisations into the shadows [30], pp. 50-51.

During World War I, production came to a halt:

This was possibly because of general disruption to all production, possibly because class sizes were bigger and the models are fragile, possibly because after equations of the fourth power that particular seam was exhausted. Fortunately they have remained on display at the Science Museum ever since to inspire new avenues of creativity [42].

They were also available in a great number of departments and museums all over the world.

The surrealist artist and photographer Man Ray (1891–1976) went to see the mathematical models at the *Institut Henri Poincaré* in Paris, and photographed them for Christian Zervos, who published an article in *Cahiers d'Art*. Some years later Man Ray used those images to create a series of paintings that he entitled *Équations shakespeariennes*.

Isabelle Fortuné writes: (Figs. 9 and 10)

Man Ray discovered the mathematical objects at the Institut Poincaré in Paris, in the company of Max Ernst, between 1934 and 1936. A collage, created by Max Ernst for an exhibition of Man Ray in November 1935, may permit us to determine the date

Fig. 9 Collection Institut Henri Poincaré. © Ghasideh Golmakani

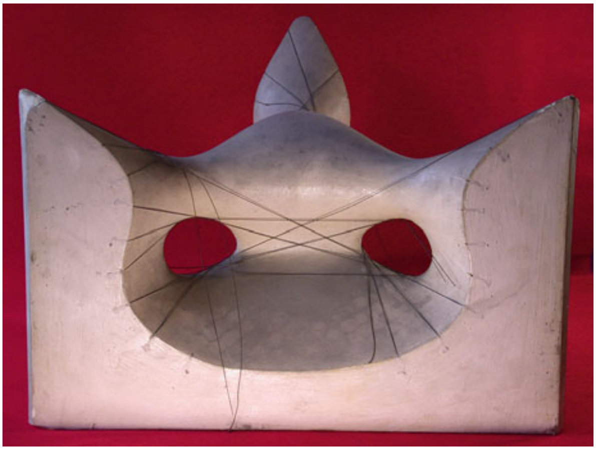


Fig. 10 Collection Institut Henri Poincaré

with greater precision (Man Ray, *Surface minimal d'Enneper*, 1934-1936; *Surface de Riemann* $w^4 = 1 - z^2$, illustration extracted from Walther Dyck, *Katalog Mathematischer und Mathematisch Physikalischer Modelle, Apparate und Instrumente*, Munich, Wolf & Sohn, 1892; Max Ernst, *Portrait de Man Ray*, 1935). He realized thirty shots, which gave rise to a dozen prints. But these few elements are the only information we have about this series of photographs. In effect, while the episode of the mathematical objects photographed by Man Ray quickly became famous in the surrealist milieu, the enthusiasm they aroused conceals a very incomplete knowledge of these photographs, although they are very frequently mentioned. In the absence of clarification by Man Ray, a legend was created about the mathematical objects. Then different versions of the circumstances of the discovery of these objects multiplied. Different assumptions of an intervention by Marcel Duchamp, of an order of Christian Zervos has been advanced, and the involvement of Max Ernst, probably decisive, is often forgotten. The relative silence that surrounds these photographs by Man Ray contributed significantly to the establishment of this legend. The occultation of this part of his photographic work is an important issue for him. Indeed, from 1948 to 1954, he painted from these photographs a series of paintings, grouped under the generic title of *Équations shakespeariennes* (Shakespearean Equations). Aspiring to a

Fig. 11 *Man Ray, Shakespearean Equation: Twelfth Night, Oil on canvas*, 1948. Hirshhorne Museum and Sculpture Garden, Smithsonian Institution. Gift of Joseph H. Hirshhorn, 1972, Photography by Lee Stalsworth

> greater recognition of his artistic talent, which he actually obtained through this series, he sought to minimize the role of their photographic antecedents. ... Taking the opposite tact to that taken by Man Ray, we will attempt to highlight the importance of the photographic medium for this series. This in effect allows us to resituate the work in the field of sculpture, which considerably increases the issues raised by this series, and makes evident the correspondence that disappears in their pictorial translation [19]. (Fig. 11)

Fortuné notes that Man Ray was quite influenced by the *Chants de Maldoror* (*The Songs of Maldoror*) by Lautréamont, especially the 'Ode to Mathematics' [25], pp. 105–108, *chant deuxième*, *strophe* 10:

> 'O stern mathematics, I have not forgotten you ... by the rigorous series of your tenacious propositions and the constancy of your iron laws, dazzle the eyes, shining forth a powerful reflection of that supreme truth whose imprint is discernible in the order of the universe.' Man Ray has frequently acknowledged the influence of Lautréamont on his work and perhaps he thought of this passage from the *Chants de Maldoror* when he discovered the

Fig. 12 *Superficie Romana di Steiner.* Mathematics Department "G. Castelnuovo", Università Roma La Sapienza

mathematical objects at the Institut Poincaré. His photographs bring to mind the cabalistic figures, equations and laconic sculptural lines sung by Maldoror. Max Ernst also makes these possible matches tangible, illustrating in 1938 the *OEuvres complètes* of Lautréamont with a collage in which figure the designs of three of these mathematical models [19].

Another multifaceted artist fascinated by mathematics and the *Chants de Maldoror* was the Italian Leonardo Sinisgalli, who studied at the Istituto Castelnuovo of the University of Rome. At the end of his first 2 years, he transferred to the Faculty of Engineering.

I was, from 1925 to 1931, a student of engineering in the buildings of Via delle Sette Sale, San Pietro in Vincoli, via Panisperna in Rome. I had distinguished teachers: first of all, Tullio Levi-Civita, then Enrico Fermi, Francesco Severi, Guido Castelnuovo, Luigi Fantappiè. I attended the two-year courses of rational mechanics; algebraic and infinitesimal analysis; analytic geometry; descriptive geometry; projective geometry; courses and laboratories in technical physics; the three years of little machines, big machines and bad machines; the lectures on metal alloys and hydrocarbons; the three-year course dedicated to strength of materials. I read the works of Archimedes, Hero of Alexandria, Leonardo da Vinci, Galileo, Torricelli, Reuleaux, Cremona, Wiener [39].

In 1936 Sinisgalli published his *Quaderno di geometria* and later *Furor Mathematicus*, of 1944. Among the topics that he dealt with, Sinisgalli wrote a short story that talks about Steiner's *Superficie romana* [30], p. 63, [37, 15, 17]: (Fig. 12)

Who would have guessed that in the shape of lupines, sufficiently enlarged, I would have one day realized Gauss's dream, the dream of a non-Euclidean geometry, or as I like to call it, a Baroque geometry, a geometry with a horror of the infinite? But just the day before

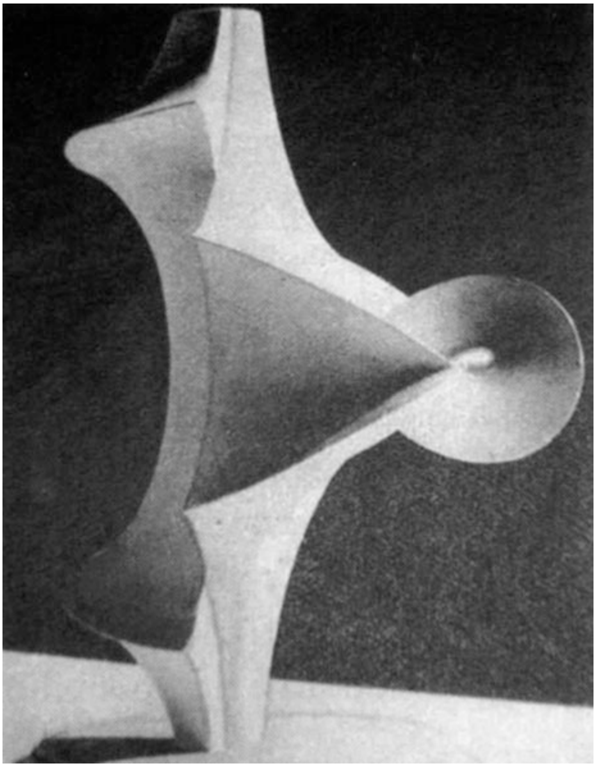


Fig. 13 Mathematical Model

yesterday I came upon a much more complex simulacrum of the shape of lupines, *Steiner's Roman surface*[3] . . .

It is a closed, complex-valued surface of degree 4. The shape I have seen is a curious one, a tuber the size of a stone, with three navels. The German mathematician Steiner discovered it while meditating one morning in the Pincian Hill, sitting on one of the exact same benches where I as a lad used to sit to read the *Songs of Maldoror*. Where had I seen the Roman surface, which was one of the models built in Germany many years earlier? [38]

Searching through my drawers I found the envelope containing the first photographic surveys that I attempted in 1945 in Rome (thanks to the kind permission of professors Fantappié and Severi of the Seminario di Matematica) of shapes in plaster, cardboard and string. I will never forget that day, and will be grateful for the rest of my life to those two *maestri* who so kindly granted my request and my explanation without thinking me mad or irreverent (Fig. 13).

[3] The well-known Steiner's Surface (or Roman Surface, so-called because according to the account given by Eugenio Beltrami (1835-1900) it was conceived by Jacob Steiner (1796-1863) during his stay in Rome in 1844) is a rational surface of degree 4 and class 3, of tetrahedral symmetry, of the Cartesian equation $x^2y^2 + y^2z^2 + z^2x^2 + xyz = 0$.

> I wanted to wrest those idols from their temple and bring them to the light of day. They did not find my proposal queer. The models were piled inside a large showcase in the antechamber of the Institute's library. You could see that they had not been taken out for many years. The dust had gathered in the folds of those sublime volumes. I called the custodian and he with great care was able to restore them to fullness without stain or wrinkle, to their timeless existence.
>
> Those small bodies, little bigger than a closed fist or pinecone, had been constructed with the Cartesian method *point by point*, beginning with an equation of x, y, z, and attributing to x and y a dual series of values of z. This method is by now familiar to the technicians who build aircraft or motors, a method of representation that transforms an algebraic expression into a form – line or surface – plane or oblique, continuous or discrete.

Sinisgalli came up with the idea of making a short film about mathematics. The 1950s were years of a great renewal of Italian cinema, and an important part of that regarded documentaries. Although the results cannot be compared to the feature films of those years which left indelible marks on cinema worldwide, some documentary shorts nevertheless were genuinely significant and opened new paths (Fig. 14).

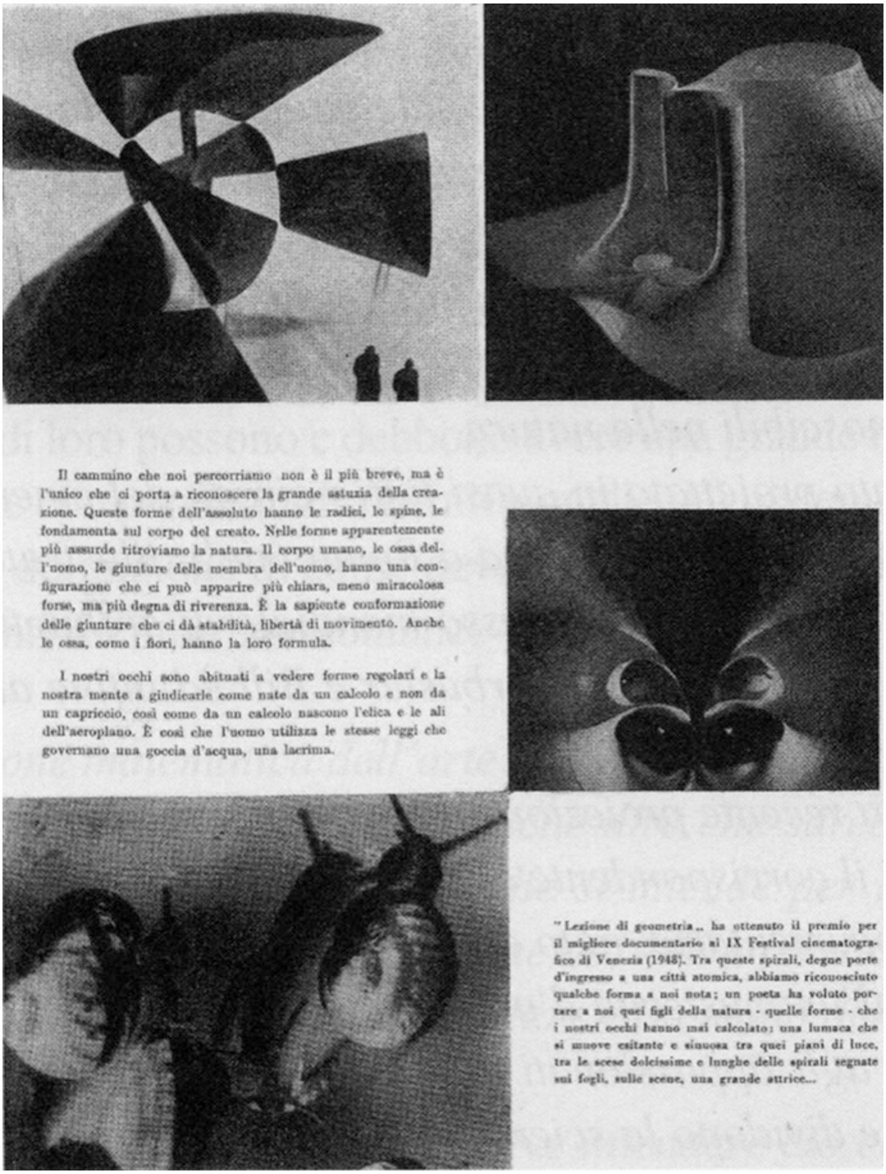
Sinigalli's interest in cinema and in documentaries were as deep as his interest in mathematics. For *Una lezione di geometria* (A geometry lesson) Sinisgalli won the prize for best Italian short film from the Ufficio Centrale di Cinematografia at the 1948 Venice Film Festival. This short, important experiment was carried out in collaboration with a unique group: director Virgilio Sabel, photographer Mario Bava, composer Goffredo Petrassi, and producer Carlo Ponti [36, 16].

Some frames from Sinigalli's film appeared in the March-April 1949 issue of the magazine *Comunità*. The film was a vehicle which allowed Sinisgalli's fascination with geometric shapes and mathematical surfaces to express itself by speaking about art, mathematics and poetry [17].

The entire documentary is narrated by a voice-over that accompanies images of the plaster models of mathematical surfaces made in Germany at the end of the nineteenth century.

Paris 1963

An unusual art exhibition took place in Paris from 20 January to 17 February 1963. In the first place the venue was unusual: the Palais de la Découverte, which was the temple to popular science in France until the *Cité des Sciences de la Villette* opened in the early 1980s. The art exhibition was entitled *Formes. Mathématiques peintres sculpteurs contemporains* (Shapes. Contemporary mathematics paintings sculptures): the title already shows that mathematics was placed on the same level as contemporary painting and sculpture [29]. The exhibition included works by very important artists, including painters Max Bill, Paul Cézanne, Robert and Sonya Delaunay, Albert Gleizes, Juan Gris, Le Corbusier, Jean Metzinger, Piet Mondrian, László Moholy-Nagy, Georges Seurat, Gino Severini, Sophie



Il cammino che noi percorriamo non è il più breve, ma è l'unico che ci porta a riconoscere la grande astuzia della creazione. Queste forme dell'assoluto hanno le radici, le spine, le fondamenta sul corpo del creato. Nelle forme apparentemente più assurde ritroviamo la natura. Il corpo umano, le ossa dell'uomo, le giunture delle membra dell'uomo, hanno una configurazione che ci può apparire più chiara, meno miracolosa forse, ma più degna di riverenza. È la sapiente conformazione delle giunture che ci dà stabilità, libertà di movimento. Anche le ossa, come i fiori, hanno la loro formula.

I nostri occhi sono abituati a vedere forme regolari e la nostra mente a giudicarle come nate da un calcolo e non da un capriccio, così come da un calcolo nascono l'elica e le ali dell'aeroplano. È così che l'uomo utilizza le stesse leggi che governano una goccia d'acqua, una lacrima.

"Lezione di geometria" ha ottenuto il premio per il migliore documentario al IX Festival cinematografico di Venezia (1948). Tra queste spirali, degne porte d'ingresso a una città atomica, abbiamo riconosciuto qualche forma a noi nota; un poeta ha voluto portare a noi quei figli della natura - quelle forme - che i nostri occhi hanno mai calcolato; una lumaca che si muove esitante e sinuosa tra quei piani di luce, tra le scese dolcissime e lunghe delle spirali segnate sui fogli, sulle scene, una grande attrice...

Fig. 14 Frames from the film *Una lezione di geometria*, Journal Comunità, March-April, 1949

Täuber-Arp and Victor Vasarely, and sculptors Max Bill, Raymond Duchamp-Villon and Georges Vantongerloo.

The exhibition was comprised of three sections, as the title indicates:

– Mathematics;

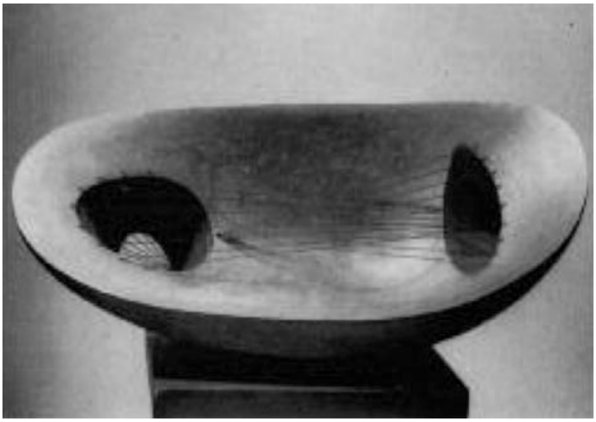


Fig. 15 Barbara Hepworth, *Landscape Sculpture.* From [29], Planche IV

– Paintings;
– Sculptures.

The first section displayed a great number of mathematical surfaces represented in metal or plaster. In particular, the following mathematical models were present:

Enneper minimal surface;
Plücker's conoid;
Möbius surface;
Möbius strip;
Klein bottle;
Cassini surface;
Surface of degree 3 with 27 real generatrixes;
Dupin cyclide;
Visualisations of four-dimensional objects.

The middle pages of the small catalogue contains four illustrations: two geometric surfaces, then the works of Robert Delaunay, Barbara Hepworth and Gino Severini. Hepworth's work of 1961 contains strings, like those of the 1930s mentioned earlier. Every detail shows that the exhibitions curators were especially careful to assign the same importance and amount of space to mathematics as to the works of the painters and sculptors (Fig. 15).

The introduction to the exhibition was written by mathematician Paul Montel, who at the time was the curator of the mathematics section of the Palais (which is, very significantly, next to the *Grand Palais,* one of the temples of art in Paris). The introduction is entitled *L'art et les mathématiques* (Art and mathematics).

> It might seem surprising that there exist relationships between art and mathematics, between the world of quality and the world of quantity. Nevertheless, close ties unite these two

> different worlds of representation ... In effect, each of these two activities, mathematical research and artistic creations, is a tributary to the other.
>
> The reason for this interdependence?
>
> An important result obtained in mathematics offers its author an aesthetic joy similar to that which is given by architectural harmony or musical consonance. We often speak of a beautiful theorem, an elegant proof. Inversely, mathematics intervenes in the conception and realisation of a work of art.

Even in the brief space of an introduction, Montel is able to provide reasons to back up his assertions:

> Beauty in architecture in particular depends on the simplicity of its proportions. Our Western eyes, for example, have been seduced for many centuries by one particular ratio, the golden section, discovered by the school of Pythagoras and transmitted by the guilds of architects and masons ...
>
> The present exhibition provides us with an opportunity to carefully examine the particular case of painting, comparing it with the mathematical objects displayed alongside.

Paris 2012

> Opening a reference point for contemporary art in the confrontation between artists and mathematicians might seem an unusual, if not incongruous, choice. This text aims to put into perspective this adventurous initiative, to illuminate the exhibition subtitle *A Beautiful Elsewhere,* and aid a better understanding of the issues of the enterprise, not at all trivial if one thinks about it.

These words are from Jean Pierre Bourguignon's introduction to the exhibition entitled *Mathématiques - un dépaysement soudain*, which took place in 2012 at the Fondation Cartier in Paris [2, 3].

A mathematical model was displayed in one room, or better, a sculptor by the Japanese artist Hiroshi Sugimoto. One part of this photographer and sculptor's work is due to the influence of mathematical models made in Germany. Those models even found their way to Japan!

> I have photographed models purchased from the West during the Meiji era (1868-1911), now preserved by the University of Tokyo. The mathematical models are sculptural renderings of trigonometric functions; the mechanical models were teaching aids for showing the dynamics of Industrial Revolution-age machinery. Art resides even in things with no artistic intentions [40].

At the beginning Sugimoto photographed those models, arriving at an effect similar to that achieved by Man Ray and Sinisgalli. He described the birth of his interest like this: (Fig. 16)

> The study of mathematics is thought have begun in ancient India and China. *Zero* and *infinity* were not so much discoveries as human inventions. The notion of length with no width is very curious indeed, the pencil line I draw being only an approximation of an

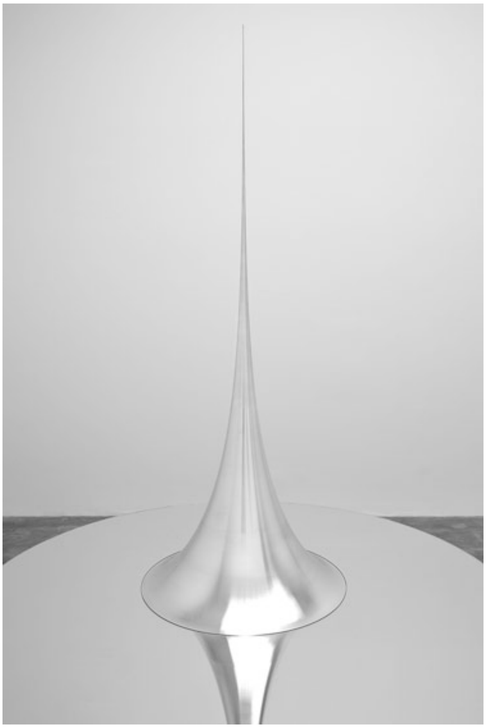


Fig. 16 Hiroshi Sugimoto, *Mathematical Model 009 Surface of Revolution with Constant Negative Curvature*, 2006. Aluminium and glass, model height: 103-1/4 in.; base diameter: 27-9/16 in.. © Hiroshi Sugimoto, Courtesy of Gallery Koyanagi

invisible mathematical line. Endeavors in art are also mere approximations, efforts to render visible unseen realms [40].

Among the notes Marcel Duchamp left in his *Green Box* are various mathematical notations. The *Large Glass* attempted to throw projections of the unseen fourth dimension onto our three-dimensional experience, much in the same way that three-dimensional objects cast shadows onto two-dimensional surfaces (Fig. 17).

While not wholly subscribing to the post-Renaissance *rational* scientific regard on the natural world, I especially appreciate those eighteenth- and nineteenth-century optical devices and experimental implements that gave visible form to unseen hypotheses [23].

He thus believed that he had to create his own mathematical models in sculpture:

During the 19th century and beginning of the 20th century, the desire to give substance to geometric ideas led to the creation of mathematical models, envisioned by mathematicians and hand sculpted in plaster by artisans.

I have attempted to transform mathematical equations into tangible objects by inputting the equations into a computer and, using Japan's most advanced machining tools, produced them in pure aluminium [40].

Sugimoto named his sculptures *Mathematical Models*, while he named his photographs of the nineteenth-century models *Conceptual Forms*.

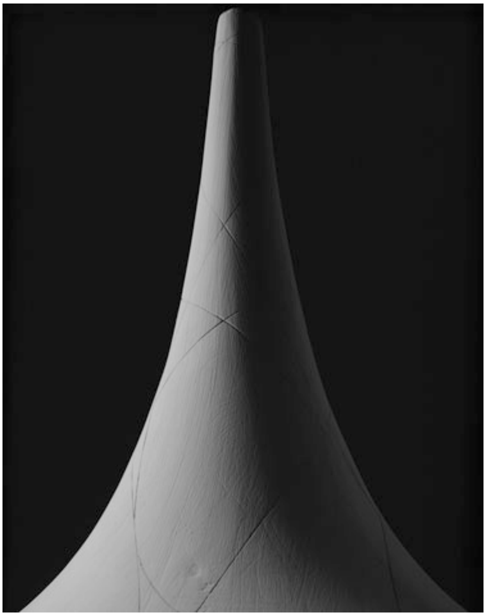


Fig. 17 Hiroshi Sugimoto, *Cenceptual Form 010.* Gelatin silver print. 58-3/4 × 47 in.. © Hiroshi Sugimoto, Courtesy of Gallery Koyanagi

Final Considerations

The history of the relationships between art and mathematics is filled with moments when this interplay aroused a great interest and others when such relationships were not noticed. There is no doubt that the period between the second half of the nineteenth century and the early years of the twentieth was an exceptionally fertile one for art and mathematics. Mathematical models played, and continue to play, an important role in this history. Born as instruments for teaching, they profoundly influenced artists all over the world. With the advent of computer graphics their history suddenly became current once again. This brief article does not tell the story of something that is irremediably in the past, but presents only a part of the ideas and interrelationships that some modern and contemporary artists wished to capture. The story deserves to be investigated in depth, and will undoubtedly produce new fruit in the field of science and art.

Translated from the Italian by Kim Williams

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Science Fiction, Art, and the Fourth Dimension

Linda Dalrymple Henderson

The notion of a higher geometrical dimension, "the fourth dimension of space," has been a vital stimulus for both writers of science fiction and artists since the late nineteenth and early twentieth century.[1] Indeed, it was science fiction writers like H. G. Wells who first responded to the new popular interest in the fourth dimension, over a decade before artists began to engage the idea. And before that, both E. A. Abbott's *Flatland* of 1884 and the "Scientific Romances" of hyperspace philosopher Charles Howard Hinton, published in the mid-1880s and mid-1890s, contributed significantly to the popularization of the idea of higher spatial dimensions. Hinton's fictional stories accompanied his even more influential treatises, *A New Era of Thought* (1888) and *The Fourth Dimension* (1904), which proposed a new philosophical system he believed could counter the evils of materialism by enabling humankind to expand its "space sense" to perceive the four-dimensional space that might hold a truer reality.

Both science fiction writers and artists would draw on this early literature on the fourth dimension, particularly the analogy of a two-dimensional Flatland in relation to a three-dimensional world, in order to reason about the relation of the three-dimensional world to a possible fourth dimension. Drawing on sources such as Hinton and many others, American architect Claude Bragdon produced in 1913 his book *A Primer of Higher Space (The Fourth Dimension)*, which codified contemporary ideas about how one might understand or visualize four-dimensional space.[2] Here we find a variety of ideas that would be paralleled in art making, beginning with the works of French Cubists in ca. 1909, as well as in science fiction writing.

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[1] This essay draws upon and expands content in the new edition of my 1983 book *The Fourth Dimension and Non-Euclidean Geometry in Modern Art* (Cambridge, MA: MIT Press, 2013) as well as the original publication. For the discussions of Hinton and Bragdon that follow, see Henderson, *Fourth Dimension*, Chaps. 1, 4.

[2] See Claude Bragdon, *A Primer of Higher Space (The Fourth Dimension)* (Rochester, NY: The Manas Press, 1913), which is now available in digital form from the Internet Archive. On the importance of the X-ray in relation to the fourth dimension, see Henderson, "Reintroduction," in *Fourth Dimension*, rev. ed. (2013), pp. 14–18.

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DOI 10.1007/978-3-319-01231-5_ 7,

Among these themes are shadows as projections from a higher dimension (with a strong connection to Plato's allegory of the cave and idealist philosophy more generally) as well as cross-sections as partial signs of higher dimensional objects. (If our world were indeed four-dimensional, we would be living in a three-dimensional section of it.) Because a four-dimensional being would see through three-dimensional objects, just as we have access to the interiors of two-dimensional objects, X-ray-like transparency also came to be associated with the fourth dimension (Fig. 1). Indeed, the discovery of the X-ray in 1895 challenged belief in the adequacy of human vision so deeply that it made it impossible to deny the existence of a fourth dimension, simply because it could not be seen.

Left/right mirror symmetry also became a standard sign of dimensional change, with the argument that a fourth dimension would be required, for example, to turn a left hand into a right hand (Fig. 2). The spiral, too, was linked to the fourth dimension because of related arguments made about left and right-handed spiraling growth in nature as well as the spiral's dimension-transcending movement. Hinton used the image of a spiral passing through a plan to explain the way in which lower-dimensional beings would misperceive a higher dimensional spiral's passage through their plane as a series of dots moving in a circle.[3] The very complexity of experiencing a higher dimensional geometry would become an important sign of the fourth dimension in both art and science fiction. The hypercube or "tesseract" would be bounded by eight cubes, as demonstrated in Bragdon's image of the folded-down hypercube (Fig. 3, Bragdon's fig. 3). In the fourth dimension, of course, it would be folded together with edges and vertices coinciding and would contain an infinite number of cubes, just as a cube itself includes an infinite number of planes. Bragdon also illustrated the hypercube in perspective projection (as one cube inside another with vertices joined), but his most effective representation of if was in isometric projection (Fig. 3, Bragdon's fig. 4). Only here do we begin to sense the spatial complexity involved, and this image would, in fact, have an important impact both on artists and science fiction writers.

This essay surveys a series of historical moments from the 1910s through the 1960s during which science fiction writers and artists responded to the theme of the fourth dimension. While for most of this period, writers and artists were responding independently from one another to the same stimuli from the realm of popularized mathematics, by the 1960s, as we shall see, an artist like Robert Smithson was actually taking stylistic clues from science fiction films, as he engaged the idea. By that time, an additional mathematical structure had been adopted by writers addressing the fourth dimension: the Moebius strip, which began to come to popular attention in the 1940s.

As noted above, the first science fiction author to respond to the newly popular fourth dimension was H. G. Wells, who published his famous book *The Time Machine* in 1895, along with two stories focused specifically on the spatial fourth dimension, "The Remarkable Case of Davidson's Eyes" in 1895 and "The Plattner

[3] See Charles Howard Hinton, *The Fourth Dimension* (London: George Allen & Unwin, 1904), Chap. 4.

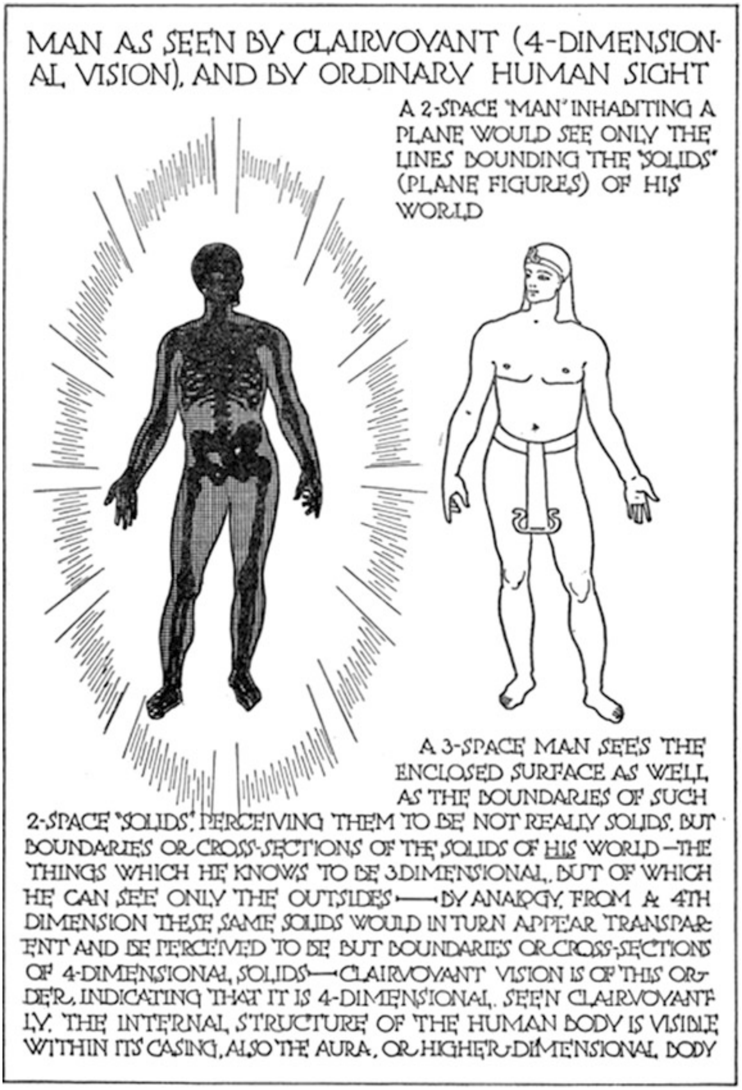

Fig. 1 Claude Bragdon, *A Primer of Higher Space (The Fourth Dimension)* (Rochester, NY, 1915), Pl. 19

Story" in 1896.[4] Unlike *The Time Machine*, these two Wells stories inaugurated the long line of science fiction tales that would utilize the marvelous effects made

[4] For these stories, see H. G. Wells, *The Country of the Blind and Other Stories* (London and New York: T. Nelson & Sons, 1913), pp. 87–100 ("The Remarkable Case of Davidson's Eyes") and pp. 204–30 ("The Plattner Story").

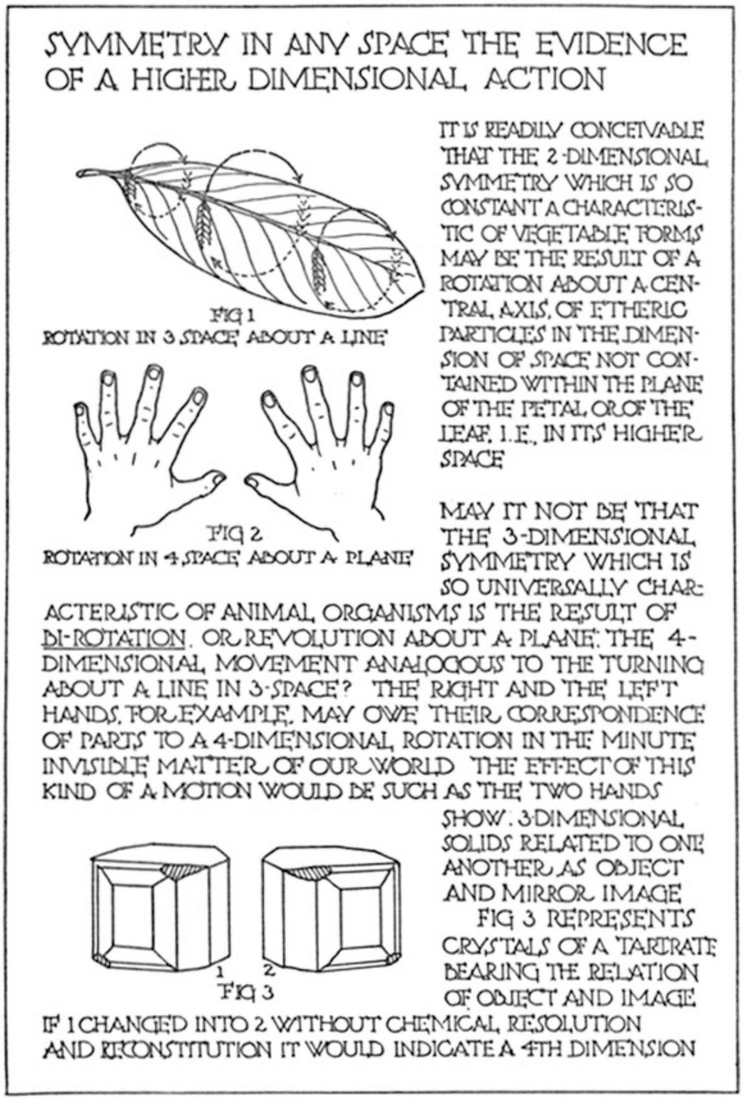

Fig. 2 Claude Bragdon, *A Primer of Higher Space (The Fourth Dimension)* (Rochester, NY, 1915), Pl. 8

possible by the addition of an extra spatial dimension. Gottfried Plattner, the unfortunate chemistry teacher of "The Plattner Story," is blown up in his laboratory and returns with his heart on the right side—in a classic fourth-dimensional mirror reversal. In "The Remarkable Case of Davidson's Eyes" a thunderous clap during an electrical storm connects the scientist Davidson in his laboratory to an island in the South Pacific. Instead of seeing the lab with all of its electrical equipment (a typical

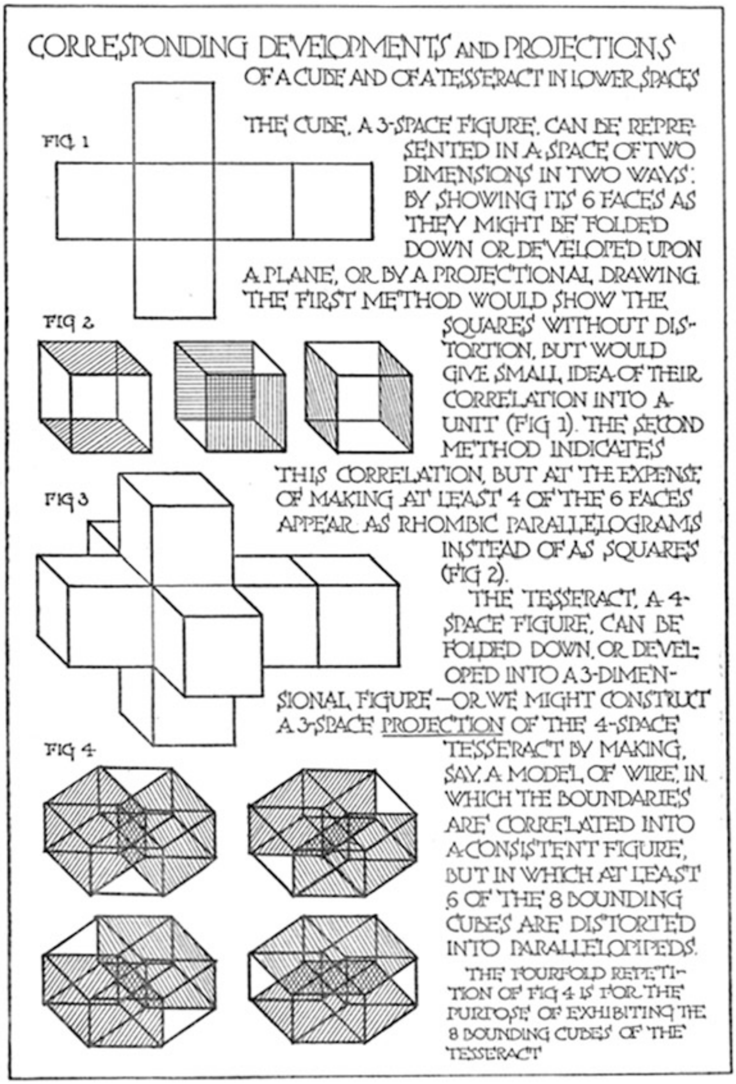

Fig. 3 Claude Bragdon, *A Primer of Higher Space (The Fourth Dimension)* (Rochester, NY, 1915), Pl. 4

setting for these transports to or through the fourth dimension), he experiences a completely different locale. For a number of weeks, he is physically present to his troubled friends in England but cannot see them or his own body. Instead, he is experiencing events in the South Sea Islands—both people and the landscape, which he "sees" vividly. However, he is in a transparent state in the South Seas, since people and other creatures simply move through his "avatar"-like being,

unaware of his presence. Davidson's transparency may well be the first usage of this phenomenon, which draws as well on the tradition of ghosts and spirits; more basic to the story is the model of space folded together through a higher dimension, connecting two far distant geographical locations—a kind of "wormhole" before the fact.

Wells's usage of time as a fourth dimension in *The Time Machine* was highly unusual in the period before the popularization of Einstein and Relativity Theory, beginning in 1919.[5] However, with his goal of writing about time travel to the future in order to make his dystopian comments on Darwinist evolution and class struggle, Wells seized upon contemporary mathematical discussions of the spatial fourth dimension to make this possible. His Time Traveler thus proposes a geometrical framework that substitutes time for the fourth spatial dimension. It is such a structure, a foretaste of Minkowski's space-time continuum of Relativity Theory, that the Time Traveler argues will make it possible to travel freely to the future. As an analogy, he cites the way contemporary balloonists have overcome gravity to be able to move in the third dimension, which had previously seemed unconquerable.

Such a spatialization of time, but with the idea of subsuming time within a fourth dimension of space was actually an element of certain discussions of the fourth dimension in this period. For Hinton, the spiral passing through a plane demonstrated the crucial, positive role time and motion could play as temporary means to understanding higher space. However, for Hinton's follower, the Russian mystic philosopher P. D. Ouspensky, time and motion as we know them are, in fact, complete illusions that will disappear once a higher consciousness of the fourth dimension is attained.[6] Indeed, the fourth spatial dimension as a sign of an unchanging eternity was also a view that emerged in this period—and it was central for the French admirer of Wells, Gaston de Pawlowski.

Cubist painters working in Paris before World War I were the first artists to attempt to evoke the spatial fourth dimension, and Pawlowski's science fiction tale, *Voyage au pays de la quatrième dimension*, serialized in the newspaper *Comoedia* primarily during 1912, served as a backdrop for their efforts.[7] Pawlowski's story contrasted the vision of a future era of idealism, the "era of the Golden Bird" in the "country of the fourth dimension," to the twentieth century's evils and illusions, based in three-dimensional materialism. He posited two successive periods, an "epoch of Leviathan" succeeded by a "scientific period," before the coming "epoch

5 See H. G. Wells, *The Time Machine: An Invention* (London: W. Heinemann, 1895), Chap. 1. On the delayed popularization of Relativity Theory, following the confirmation of one of Einstein's postulates by an eclipse expedition in November 1919, see, e.g., Helge Kragh, *Quantum Generations: A History of Atomic Physics in the Twentieth Century* (Princeton: Princeton University Press, 1999), pp. 98–104.

6 For Ouspensky's philosophy, see Henderson, *Fourth Dimension*, Chap. 5.

7 Gaston de Pawlowski, "Le Léviathan," *Comoedia*, Dec. 24, 1909, p. 1. The following discussion is drawn from Henderson, *Fourth Dimension*, rev. ed. (2013), pp. 151–56, 466–67. Chapters of Pawlowski's *Voyage* were published over the course of 1909–1910 and 1912 in different series in *Comoedia*. His book of this title appeared in later 1912 (Paris: Eugène Fasquelle, 1912) and was published in a revised edition in 1923 (Paris: Eugène Fasquelle, 1923; Paris: Denoël, 1971). Pawlowski's statement about "all phenomena and their opposites" in the next paragraph occurs in the 1923 revised edition, pp. 15–16.

of the Golden Bird" in 3789 (2000 years after the declaration of the Rights of Man in 1789). A great admirer of H. G. Wells, Pawlowski, however, accomplished his travel to the various epochs without a time machine like that of Wells. Instead, he did so by means of what he terms "abstractions of space" and "abstractions of time," in which instantaneous voyages to the future occurred.

Devoted to the spatial fourth dimension, Pawlowski mixed Wellsian social and political critique with an idealist hyperspace philosophy akin to that of Hinton. According to Pawlowski, introducing his serial in 1909, the fourth dimension could be thought of as "a different point of view, a manner of envisaging things in their eternal and immutable aspect. For Pawlowski, the fourth dimension represented the "sole contact between the eternal Idea and matter" as well as "the explanation of all phenomena and their opposites"—i.e., an absolute, timeless reality that would resolve all contradictions apparent in the three-dimensional world. In the "country of the fourth dimension," time as succession would be completely replaced by the true simultaneity of all existence, much as Ouspensky argued in *Tertium Organum.*

Although idealist philosophy would play a role in Cubist theory, most Cubist painters were too interested in Henri Bergson's vision of flux and continuity to paint still, timeless images or to believe in any single, absolute Platonic form.[8] Instead, Cubist paintings attempt to evoke a complex, four-dimensional reality beyond vision that drew as well on contemporary reconceptions of matter and space in the wake of X-rays, radioactivity, and the focus on vibrating ether waves of wireless telegraphy. As I have discussed elsewhere, both in Cubist paintings, such as Jean Metzinger's *Dancer in a Café* (Fig. 4), and in French geometer E. Jouffret's 1903 rendering of a "see-through" view of a four-dimensional solid, transparent, multiple views of an object as well as shifting, shaded facets create an ambiguous space that cannot be read as three dimensional (Fig. 5). "I paint objects as I think them, not as I see them," the pioneering Cubist Pablo Picasso declared, and—along with his engagement with the art of Cézanne and African sculpture—contemporary interest in a higher dimension of space encouraged his increasingly conceptual approach to the visible world. Cubist theorists Albert Gleizes and Metzinger drew directly on Henri Poincaré's ideas on tactile and motor sensations in his 1902 *La Science et l'hypothèse*, where he asserted that "motor space would have as many dimensions as we have muscles" and suggested that one might represent a four-dimensional object by combining multiple perspectives of it.

If Cubist paintings succeed in suggesting four-dimensionality, it is because they deny being read as three dimensional. Gone is Renaissance perspective in favor of geometrical complexity. Nor is there any longer a single light source modeling forms in visible light. Rather, shifting, transparent forms suggest the multiple viewpoints deriving from Cézanne and augmented by Poincaré. As noted earlier, Cubist theory also had a strongly idealist orientation, which paralleled Pawlowski's

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8 The discussion of Cubist painting in the following paragraph parallels the argument I have made in Henderson, "Editor's Introduction II. Cubism, Futurism, and Ether Physics in the Early Twentieth Century," in *Science in Context*, 17 (Winter 2004), pp. 445–66.

Fig. 4 Jean Metzinger, *Dancer in a Cafe*, 1912. ©ARS, NY. Photo Credit: Albright-Knox Art Gallery, Buffalo, New York/Art Resource, NY

anti-materialist stance, if not his embrace of a changeless eternity in the fourth dimension.

Moving forward in time, the 1920s would see the beginning of an entire genre of fourth dimension-related science fiction, which would thrive in the 1930s, 1940s, and into the 1950s, producing a range of stories by authors from Lewis Padgett to Arthur C. Clarke.[9] The decades of the 1920s–1940s were the period when the highly popular concept of the "fourth dimension as space" was gradually eclipsed

9 See, e.g., Lewis Padgett, "Mimsy were the Borogoves," in *A Treasury of Science Fiction*, ed. Groff Conklin (New York: Crown Publishers, 1948); and Arthur C. Clarke, "Technical Error," in *Reach for Tomorrow* (New York: Ballantine Books, 1956).

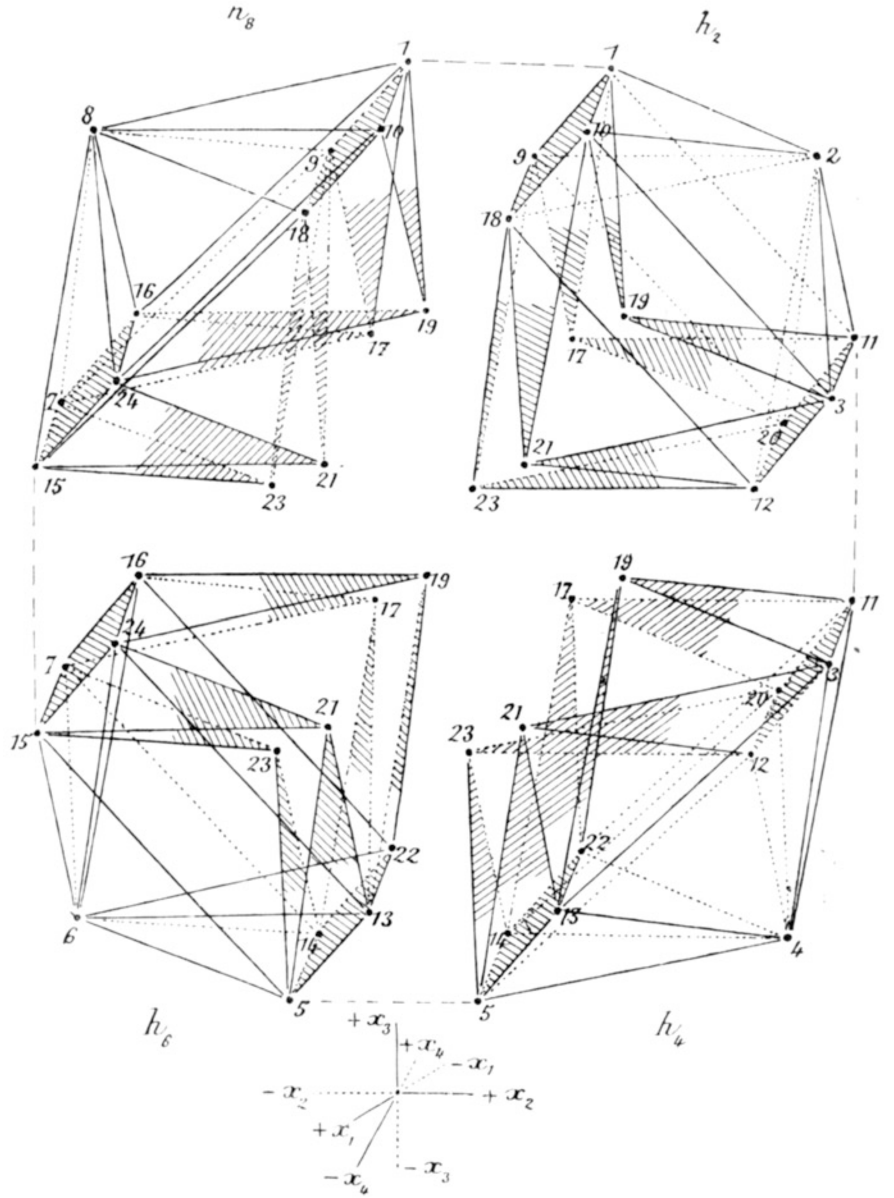


Fig. 5 "*Perspective cavalière* of the Sixteen Fundamental Octahedrons of an Ikosatetrahedroid," from E. Jouffret, *Traité élémentaire de géométrie à quatre dimensions*, Paris, 1903, Fig. 41

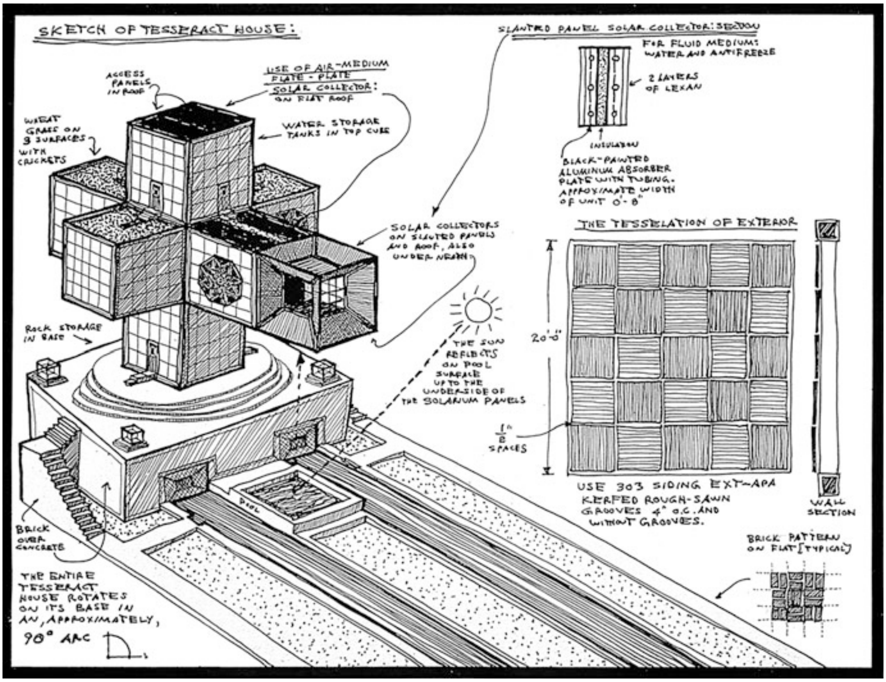


Fig. 6 Paul Laffoley, *Tesseract House Project* (one of 18 designs), 1978. Courtesy of Kent Gallery, New York

by the growing dominance in the public's mind of the temporal fourth dimension of Relativity Theory's space-time continuum. In fact, science fiction served as one of the most important "underground" threads that kept interest in the spatial fourth dimension alive in these decades, along with the continued presence of *Flatland* and mystically oriented literature, such as the texts of Bragdon and Ouspensky, which remained in print.

One of the greatest of all the science fiction stories devoted to the fourth dimension is Robert Heinlein's "—And He Built a Crooked House" of 1941. Heinlein's tale may well be the most effective evocation ever written of the paradoxical effects that would result in the course of navigating through a four-dimensional object. Artist Paul Laffoley, trained as an architect, was so intrigued by Heinlein's story that in 1978, many years after he initially read it, he produced a detailed set of plans for Teal's unfolded "Tesseract House" (Fig. 6).

In Heinlein's story, after a California earthquake, architect Quintus Teal returns with his clients to the site of a house he had built them in the form of an unfolded hypercube. Unbeknownst to the mystified Teal, the house had been folded into the fourth dimension during the earthquake, producing an effect far different that the rectilinear order of Laffoley's drawing of the pre-folded house. Teal declares to his clients, who have made their way into the house, but can no longer find the exit door:

> "Now here's what we have to do: As I see it, in a four-dimensional figure a three-dimensional man has two choices every time he crosses a line of juncture, like a wall or a threshold. Ordinarily he will make a ninety-degree turn through the fourth dimension, only he doesn't feel it with his three dimensions. Look." He stepped out of the very window he had fallen out of a moment before. Stepped through and arrived in the dining room, where he stood, still talking.
>
> "I watched where I was going and arrived where I intended to .... The time before I didn't watch and I moved on through normal space and fell out of the house. It must be a matter of subconscious orientation."[10]

Subsequently, Teal and his clients discover even stranger hyperspatial effects. Spotting a man several rooms away, Teal pursues him, only to discover from the man's dropped hat, bearing the initials Q. T., that it is himself. Finally, upon opening successive window curtains in the study at the center of the house, they see New York as if from the top of the Empire State Building, an inverted seascape with waves rolling overhead, complete nothingness, and finally a desert into which they hurl themselves, having had their fill of "space folded over through the fourth dimension ... and looking past the fold."

While Heinlein here echoes the theme from Wells's "Remarkable Case of Davidson's Eyes," involving distant geographical locales folded together, he was most inventive in his description of the experience of moving from room to room within the house. We are reminded of that spatial complexity by the isometric projection of the hypercube modeled by sculptor Peter Forakis in his *Hyper-Cube* of 1967, for which he drew directly on Bragdon's writings and illustrations (Fig. 7).[11] Heinlein effectively captures that complexity in prose as he describes the paradoxes inevitable for anyone grounded in three-dimensional cognition, such as Teal and his clients. In fact, paradox itself would come to be associated with the fourth dimension in popular writing on the subject. Ouspensky, for example, had preached the need to develop a system of seeming alogical logic in order to free the mind from the constraints limiting its access to higher dimensional understanding. This paralleled Hinton's original call to develop one's "space sense" by overcoming the sense of gravity and left/right orientation that constrain us in three-dimensional space.[12]

For Wells, the idea of time as a dimension easily traversed—forward and backward like the balloonist's ascent and descent—had offered free passage and a powerful demonstration of free will and choice. For writers in the period of Relativity Theory's population after 1919, the space-time continuum likewise treated time

[10] Robert Heinlein, "—And He Built a Crooked House," in *Fantasia Mathematica*, ed. Clifton Fadiman (New York: Simon and Schuster, 1958), p. 83.

[11] For Forakis's sculpture and its relation to Bragdon, see Henderson, "Reintroduction," to *Fourth Dimension* (2013), pp. 61–63; and Linda Dalrymple Henderson, "Claude Bragdon, the Fourth Dimension, and Modern Art in Cultural Context," in *Claude Bragdon and the Beautiful Necessity*, ed. Eugenia Ellis and Andrea Reithmayr (Rochester, New York: Rochester Institute of Technology, 2010), pp. 61–63.

[12] On these approaches, see Henderson, *Fourth Dimension*, Chaps. 1, 5.

Fig. 7 Peter Forakis, *Hyper-Cube*, 1967. Aluminum, $35\frac{7}{8} \times 36\frac{7}{16} \times 36\frac{1}{8}$ in. Walker Art Center, Minneapolis, Gift of Virginia Dwan

as a geometrical dimension; however, that structure, unlike Wells's fantasy, now created a stricter interconnection between spatial and temporal events. Nonetheless, the idea of a "block universe" offered the image of a past continuing to exist and a pre-existent future, both of which might be visited if one could gain access to another, fifth dimension outside the block universe. That was the argument of the highly influential British aeronautical engineer J. W. Dunne, whose 1927 book A *Experiment with Time* suggested that in dreams such an extra-dimensional foresight of the future might be possible. Once again, it was dimensional augmentation that would provide a new kind of experience of the ordinary world.[13]

The idea of a block universe, albeit twisted in the form of a Moebius strip, was central to Mark Clifton's 1952 story "Star, Bright."[14] Clifton's story centers on an unusually bright three-year old girl named Star, whose remarkable intelligence regularly startles her father. After starting nursery school at age four, Star explains to her father that most people are what she calls "Stupids," she is a "Bright, and he is a "Tween" (for in-between). At age three she had already constructed a Moebius strip and seemed deeply interested in its properties, which included the move from a flat, two-sided surface to a twisted, no-sided surface requiring an extra dimension to contain it. Having entered science fiction in the 1940s from the realm of topology, Moebius strips were often utilized as vehicles for transcending dimensions, as in A. J Deutsch's "A Subway Named Moebius" of 1950.[15]

Not only had Star mastered reading at age three, but she also exhibits telepathic powers, regularly reading her flustered father's mind. When another "Bright,"

13 See J. W. Dunne, *An Experiment with Time* (New York: The Macmillan Co, 1927).

14 See Mark Clifton, "Star, Bright," in *The Mathematical Magpie*, ed. Clifton Fadiman (NewYork: Simon and Schuster, 1962), pp. 70–96.

15 See A. J. Deutsch, "A Subway Named Moebius," in *Fantasia Mathematica*, ed. Clifton Fadiman (New York: Simon and Schuster, 1958), pp. 222–36.

6-year old Robert, moves in next door, they become regular playmates. Robert turns out to have telepathic powers as well, and the two children develop quite unusual games. In their version of hide-and-seek, one child stands in front of a bookshelf of encyclopedias, and then, as Star explains to her father, "We just TP [teleport] ourselves by ESP anywhere we want to go."[16] That they have traveled to ancient Rome and Egypt is demonstrated when Star accidentally drops a brand-new coin from one of the ancient realms they have visited.

Star and Robert accomplish their teleporting by extending the principle of the Moebius strip to three and four dimensions. As the 4-year old Star explains, "Next you take a sheet and you give it a half twist and join the edge to itself all over to make a funny kind of holder"—Klein's bottle, the three-dimensional analog in topology to the Moebius strip.

"Well, then, the next step—you take a cube." "You can't do this with your hands. You've got to ESP it done, because it's an imaginary cube anyway." ...

"And you ESP the twisted cube all together the same way you did Klein's bottle. Now if you do that big enough, all around you, so you're sort of half-twisted in the middle, then you can TP [teleport] yourself anywhere you want to go. And that's all there is to it ... ."[17]

When Star's father asks them about whether they've travelled to the future, he is shocked by Robert's response: "There isn't any future .... Everything is always past." Robert has discerned that the block universe has itself been twisted like a Moebius strip, so that the succession he details—"Cave Men, This Men, That Men, Mu Men, Atlantis Men, Egyptians, History Men, Us Now Men, Atom Men, Moon Men, Planet Men, Star Men," repeats itself as "Cave Men" follow immediately "Star Men." "It's all one thing joined together," he continues. "It isn't future and it isn't past either. It just plain *is*."[18]

With Star and Robert's clues about teleporting, Star's father gradually learns to travel to various moments in the seeming "past." But, in the meantime, the children have become fascinated with how to teleport themselves off the Moebius strip-like universe into another present. When one day, the children vanish and do not return in the customary hour or two, Star's panicked father decides he must follow them, and he begins to reason with his limited "Tween" brain: Considering that a twisted cube made it possible to teleport anywhere on the Moebius-strip-like universe, he continues, "Let's go a step higher, into one more dimension. In short, the tesseract. To get the equivalent of a Moebius strip with depth you have to go into the fourth dimension, which, it seems to me, is the only way the Bright Ones could get off this closed sycle of past-present-future. They must have reasoned that one more notch up the dimensions was all they needed."[19]

16 Clifton, "Star, Bright," p. 85.
17 Ibid., p. 86.
18 Ibid., p. 89.
19 Ibid., p. 95.

The story concludes with Star's father penning a note of explanation to a scientist friend and Robert's parents about his plan to follow the children off the Moebius strip universe. He writes,

"I'm trying now to take six cubes and fold them in on one another so that every angle is a right angle. It's not easy, but I can do it, using every bit of concentration I've learned from the kids. All right, I have the six cubes and I have every angle a right angle. Now if, in the folding, I ESP the tesseract a half twist around myself and—"[20]

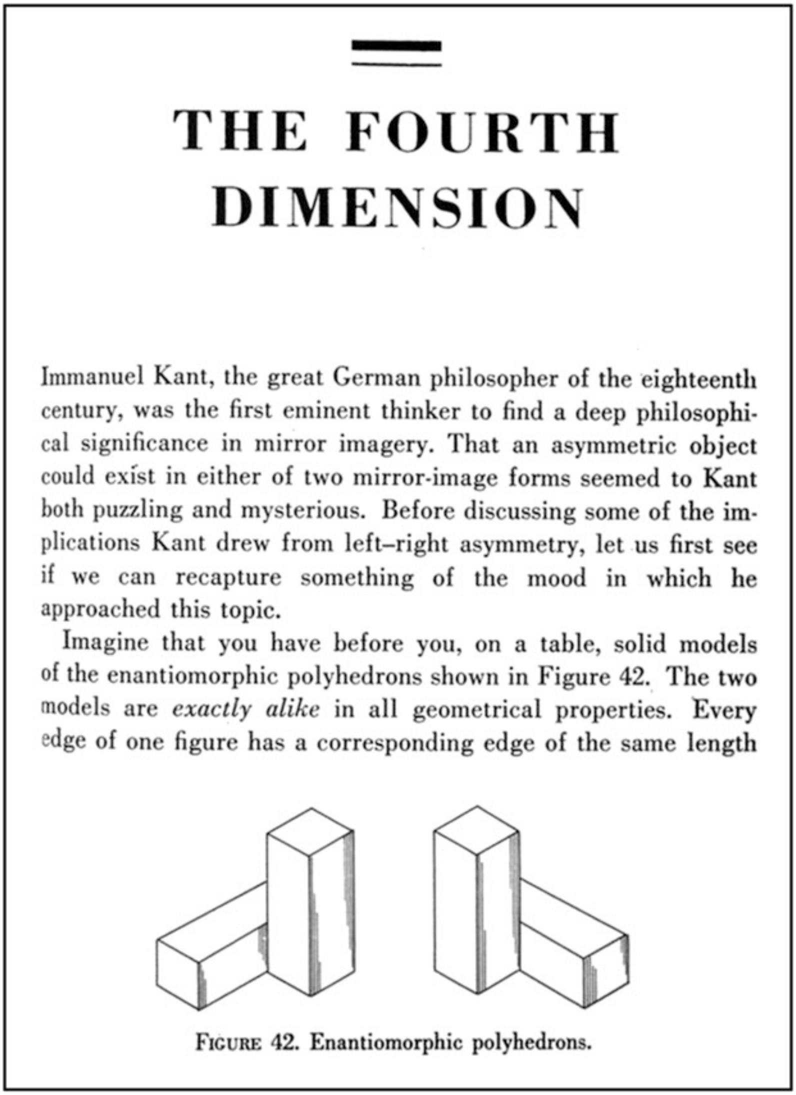
Clifton's story managed to fuse Einsteinian space-time as a block universe, the Moebius strip, and the tesseract or hypercube of Hinton and Bragdon from the heyday of the spatial fourth dimension in the late 19th and early twentieth century. The spatial fourth dimension was definitely coming back, a process that would be accelerated in the later 1950s and early 1960s at the hands of Martin Gardner in his *Scientific American* "Mathematical Games" columns and his books, such as *The Ambidextrous Universe* of 1964. Robert Smithson was among the artists who responded enthusiastically to the "new"/old fourth dimension. Gardner's *Ambidextrous Universe*, with its chapter on "The Fourth Dimension," was a direct source for Smithson's *Enantiomorphic Chambers* of 1965, which included mirrors and explored left-right mirror reversals on the model of Gardner (Fig. 8).[21] Smithson's *Chambers* strongly suggest the form of the right- and left-handed crystals he actually cut out of Gardner's chapter, similar to Bragdon's illustration in the *Primer of Higher Space* (Fig. 2), where he had asserted that "If 1 turned into 2 without chemical resolution and reconstitution it would indicate a 4th dimension."

Our "time travel" from the 1910s to the 1960s concludes with another series of works by Smithson, his *Mirrored Vortices* of 1965. These sculptures continued the interest in mirror reflections and their four-dimensional associations he had found not only in Gardner but also in the art and writings of Marcel Duchamp, the early twentieth-century artist most deeply engaged with four-dimensional geometry.[22] Here Smithson created a highly complex spatial experience for a viewer, who looks down into an angular "well" in which three or four mirrors reflect one another as well as elements of the room, creating an extremely disorienting interpenetration and repetition of images. For Smithson and his friend artist Peter Hutchinson, such works bore additional associations with higher dimensions through the connections they perceived to science fiction. Hutchinson's 1968 article "Science Fiction: An Aesthetic for Science," had compared the work of a number of contemporary sculptors to the "purposeless" machines on a science fiction film set. Sensing a shared focus on "surface" in science fiction and the new sculpture, Hutchinson notes, "Science-fiction has been fascinated with crystallography, topology, outrageous color, extreme

20 Ibid., p. 96.

21 For Gardner's *Scientific American* articles and book as well as Smithson's response to it, see Henderson, "Reintroduction" to *Fourth Dimension* (2013), pp. 50–55, 58–61. Images of Smithson's sculptures are readily accessible at www.robertsmithson.com.

22 On Smithson and Duchamp, see Henderson, "Reintroduction" to *Fourth Dimension* (2013), pp. 57–58; on Duchamp and the fourth dimension, see ibid., pp. 38–42 and Chap. 4.



THE FOURTH DIMENSION

Immanuel Kant, the great German philosopher of the eighteenth century, was the first eminent thinker to find a deep philosophical significance in mirror imagery. That an asymmetric object could exist in either of two mirror-image forms seemed to Kant both puzzling and mysterious. Before discussing some of the implications Kant drew from left–right asymmetry, let us first see if we can recapture something of the mood in which he approached this topic.

Imagine that you have before you, on a table, solid models of the enantiomorphic polyhedrons shown in Figure 42. The two models are *exactly alike* in all geometrical properties. Every edge of one figure has a corresponding edge of the same length

FIGURE 42. Enantiomorphic polyhedrons.

Fig. 8 Martin Gardner, *The Ambidextrous Universe* (New York: Basic Books, 1964), p. 161

geometric form, new materials, as well as concepts like space warp, time travel, extra dimensions. These themes have remarkable counterpoints in current work by many artists."[23] Throughout his essay Hutchinson quotes repeatedly from Norman Spinrad's 1964 story "The Rules of the Road," which centers on the hero's

[23] Peter Hutchinson, "Science Fiction: An Aesthetic for Science," *Art International*, 12 (Oct. 1968), 33.

experience of transport into a higher dimension. The passages he selects also evoke contemporary Minimalist sculpture:

> He was enclosed in a cubical box of some dull metal . . . . He was trapped. He was surrounded on all six sides. But in this space a cube did not have six sides, it had thirty-six . . . . He moved at right angles to all six faces of the contracting cube simultaneously. He was out and he was in a space with no dimensions. He was every point in the space, since all points coincided.

Ultimately, the hero discovers that higher dimensional space "is the real universe" and that "[a]ll else is illusion, a partial truth, the projection in three dimensions of a reality with an infinity of dimensions . . . ."[24]

Examining the synergy among the popular fourth dimension, science fiction, and art augments considerably our understanding of the way in which a mathematical concept like the spatial fourth dimension came to function so prominently in twentieth-century culture. When in 1953 the House of Gourielli marketed its perfume called "Fourth Dimension," it clearly relied visually on the popular association that had developed by this time between modern art and the fourth dimension. But for its advertising copy, the Gourielli firm utilized ideas about the spatial fourth dimension that had been kept alive primarily in science fiction, such as "freedom of motion," "freedom of spirit," and "unlimited . . . possibilities." [25] Soon the Space Age itself would realize certain of those possibilities, including ideas long present in writing on the fourth dimension, such as the vastness of space and, specifically, the freedom from gravity and orientation that Hinton had touted. It is little wonder then that Arthur Clarke's screenplay for the film *2001* actually included plans for a tetrahedron through which the astronauts would look and "see into another dimension."[26] But the age of computer-generated special effects had not yet arrived, and such suggestions of extra-dimensional experience would have to wait until the later 20th and 21st centuries.

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[24] Ibid, pp. 33–34.

[25] See, e.g., *The New Yorker*, 29 (Dec. 5, 1953) for an image of a later Gourielli "Fourth Dimension" ad (1955), see Henderson, "Reintroduction" to *Fourth Dimension*, rev. ed. (2013), p. 11.

[26] For this Doug Trumbull comment, quoted in Gene Youngblood's 1970 book *Expanded Cinema*, see Ibid, p. 66.

Donald Judd's Arithmetics and Sol LeWitt's Combinatorics. On the Relationship Between Visual and Mathematical in New York Art Around 1960

Michael Rottmann

The *Serial Project # 1* as Visual Paradox

In 1966 Sol LeWitt exhibited his *Serial Project #1 (ABCD)* at New York's Dwan Gallery (Fig. 1). Just a few months later, in April 1967, in her essay entitled 'Sol LeWitt: The Non-Visual Structures', the art critic Lucy Lippard had diagnosed what seemed to her to be a paradoxical relationship in his art between the visual and meaning, and called this a main theme in LeWitt's creative activity [29, p. 42ff]. For Lippard it was contradictory, because—according to LeWitt's own words—the appearance of a work of art, although it is visual art, is not relevant and could in fact in principle be arbitrary, but nevertheless the underlying combinatorial rule can be conveyed precisely.[1]

The *Serial Project #1* is a good example of how LeWitt coupled his works of art in the 1960s to the 'mathematical', to deal with questions of visuality. The term 'the mathematical' is used to denote that mathematics was converted by the artists within the arts. Examining these historical results, which are interesting for visual sciences as well, this paper will—on the basis of two examples of work by Sol LeWitt (1928–2007) and Donald Judd (1928–1994)—explain a debate in the art circles in New York of the 1960s concerning aspects of the visual in the production and reception of art, with the artists' own remarks and works and the manner on which mathematics or the mathematical intersected with this. In short, mathematics respectively the 'mathematical' gave rise to conceptions of seeing as well as the experience and meaning of works of art. First we will look at Judd's work *Untitled* 1965 (DSS 60) from the series entitled *Progressions*, which LeWitt himself has said provided the stimulus for his *Serial Project #1*.

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This essay is written in context of my doctoral thesis 'Geometrien, Zahlen, Diagramme. Die New Yorker Kunst um 1960 im Spiegel der Mathematik' (Geometries, Numbers, Diagrams. Fine Art in New York Around 1960 in the Mirror of Mathematics) submitted at the Freie Universität Berlin.

[1] It was traditionally believed that the concrete material execution and the meaning had to be firmly coupled.

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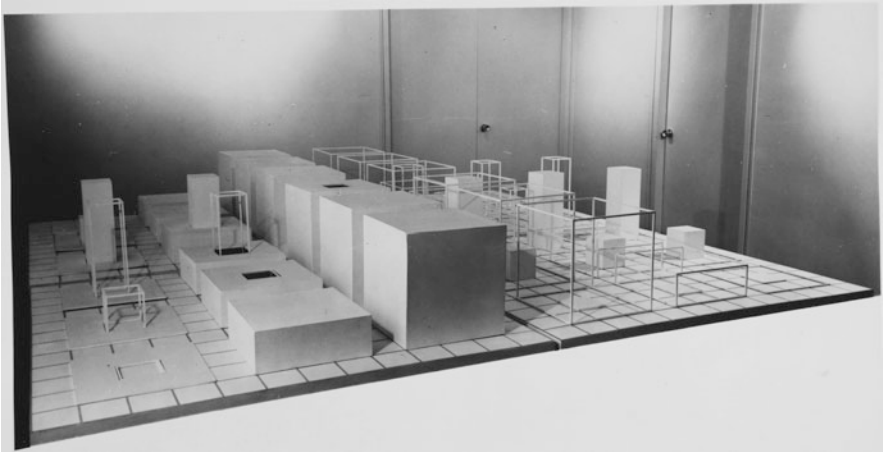

Fig. 1 An installation view of Sol LeWitt's *Serial Project #1 (ABCD)* (1966) at the Dwan Gallery in New York, February 1966/unidentified photographer. Dwan Gallery records, Archives of American Art, Smithsonian Institution. Sol LeWitt, *Serial Project #1 (ABCD)*, 1966, burnt enamel on steel, 51 × 200 × 200 cm. Licenced by VBK Bildrecht, Wien 2014

Numeric 'Schemes' and the Visual in Donald Judd

In 1964 Judd began to work on the *Progressions*. As shown in *Untitled* 1965 (DSS 60) it consists of stripes which are rounded off on the front or cuboid-shaped, often having an extreme elongated format and characterized particularly by the rhythmical division of positive and negative volumes (Fig. 2).[2] If Judd determined the measurements of three-dimensional volumes with numbers as in the progression, then this procedure can be understood as a contribution to the discourse of visuality. Judd's manner of numeric production was tied to the aim of contrasting the European compositional practices of the Renaissance and geometric abstraction, both of which appear as gradual processes aligned with the visible work of art. These can be traced back to Leon Battista Alberti (1404–1472) and Piet Mondrian (1872–1944), both of whom are mentioned frequently by Judd.

While for Alberti, according to his famous definition, harmony and beauty in a composition were achieved when nothing could be added or taken away [2, p. 293], Mondrian was interested in creating a balance of the elements in his paintings by intuitively arranging them with the intention of representing universal order. According to Judd, what was operating in these two cases was '[t]ypical part-by-part play' [9, p. 47]. This characterization of the production manner can be seen as a criticism of the importance attributed to the visual, since the elements

[2] With the broad format Judd wanted, among other things, to dissociate himself from the painting and its typical formats.

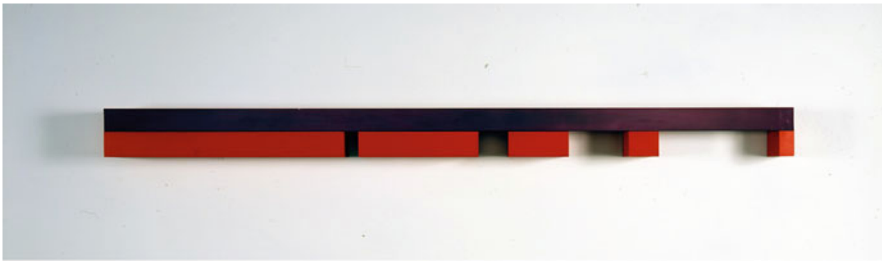

Fig. 2 Donald Judd, *Untitled* 1965 (DSS 60), violet lacquer on aluminium and light cadmium red enamel on aluminium, 13.4 × 189.7 × 13.3 cm, Collection of Phoenix Art Museum, Phoenix, Arizona. Gift of Howard and Jean Lipman 1983.109. Licensed by Bildrecht, Wien 2014. Photo ©Phoenix Art Museum, Phoenix, Arizona

of a work of art which would be added are ordered in coordination with what is already visible. Instead of carrying out a process of feedback, as the examining and correcting reflexive interaction of eye and hand could be described, in keeping with the cybernetics of Norbert Wiener in the 1960s, Judd determined the lengths of forms and distances in advance, as in *Untitled* 1965 (DSS 60), using number sequences. The artist explained it this way: 'So it's not conceived part by part, it's done in one shot. The progressions made it possible to use an asymmetrical arrangement, yet to have some sort of order not involved in composition' [9, p. 47ff]. The criticism might be raised that Judd also composed his works, if he designed them first in drawings on paper listing the measurements, and could anticipate the definite built works by the collected experience of implemention. However, there is a qualitative difference. In his drawings the artist carried out acts of substitution; in the actual work of art which was manufactured at that time he didn't join any parts together. The fact that the visual impact of a built work of art could be anticipated, but not actually be represented in any other medium, such as a drawing, was treated as part of the ongoing discourse on the visual.

In the famous 1964 interview with Bruce Glaser, Judd, like his artist colleague Frank Stella, explicitly pointed out the difference between the medium of drawing and the accompanying work and the necessity of translating the first into the second: 'One thing I want is to be able to see what I've done, as you [Frank Stella] said. Art is something you look at' [17, p. 164]. These considerations were not exclusively based on the visual: Robert Morris, from 1966 onwards, considered the work-body-relationship as a criterion of non-visual art perception according to Husserl's phenomenological concept of corporality (*Leiblichkeit*).

This isn't the place to go into detail about the obvious discrepancies between Judd's anti-compositional conception and his artistic practice [32, p. 149; [18], p. 456]. Let us instead examine the visual effect of the works. Judd criticized not only European production methods but also the works of art resulting from them. To Judd, neither beauty by balanced compositions according to Alberti, nor balance like that in the relational art of Mondrian were worth striving for. Instead, he

pursued ‘wholeness’ [17, p. 155], which was difficult to realize. Ideally, a work of art shouldn’t be taken to pieces in contemplation, and should appear as a holistic object, as a visual entity. Moreover, the parts and the whole should be related to each other in equal relations, with the parts neither emphazised nor subordinated and arranged hierarchically to achieve a higher-order whole-like in a figurative sculpture or in a painting which represents the universal, as in the compositions of Alberti and Mondrian. With his number series, which he called ‘schemes’, the artist could create asymmetrical divisions in his works. Although Judd designed the *Progressions* as relatively compact, their aesthetically fascinating formal complexity was again and again in tension to the *wholeness*. The more complex the organisation of a work, the greater the risk of its disintegrating into parts. Another related problem existed: although Judd’s works are not composed (in Alberti’s sense of the term), they have an order. Again in 1964, still under the fresh impression of the production of his early box-works, the artist explained it in the interview with Bruce Glaser:

> Take a simple form – say a box – and it does have an order, but it’s not so ordered that that’s the dominant quality. The more parts a thing has, the more important order becomes, and finally order becomes more important than anything else [17, p. 156].

Presumably Judd had noticed this phenomenon in the box-work *Untitled* 1963 (DSS 41), in which the order of the lamellae is undoubtedly an important element (Fig. 3). As the narrowing distances of the lamellae are determined according to a number sequence [1, p. 42], a problem of the numeric production—which began with *Untitled 1963* (DSS 41)—can be identified: the order brought an increase of significance, because their schemes could come into the focus of the observer and be identified if necessary, because the schemes as factors of production were, so to speak, ‘embodied’ in the works. And in contrast to an anthropomorphic Renaissance sculpture, in which proportions are embodied in a body, no pictorial effect or beauty in Alberti’s sense detracts the viewer from the numeric design in Judd’s Progressions. This could give rise to various effects in which the artist had no interest. There arose a competitive relationship between the visible work and its intelligible part. This contradicts Judd’s interest in non-referential works, where semantically neutral properties of the material, the colors and shapes should be presented; this is why, incidentally, the artist used primary geometric shapes. Judd could not escape the mathematical connotation of his works and the assignment of his oeuvre in proximity to mathematics, which occurred from 1966 at the latest, as the history of the reception of his art demonstrates.[3] He opposed the unwanted equating of his art with proportional aesthetics because, among other things, of its metaphysical charging and its anachronistic character. Judd responded by not always using a universal order, such as the golden section, but different and individual number sequences, his ‘schemes’. Alberti had demanded that, particularly for the

[3] The application of arithmetic had been discussed in 1966 in reviews, in articles by Robert Smithson, Mel Bochner Lucy Lippard and Rosalind Krauss, and in the following years by art critics such as Elizabeth C. Baker, John Coplans and William C. Agee. See for example [8].

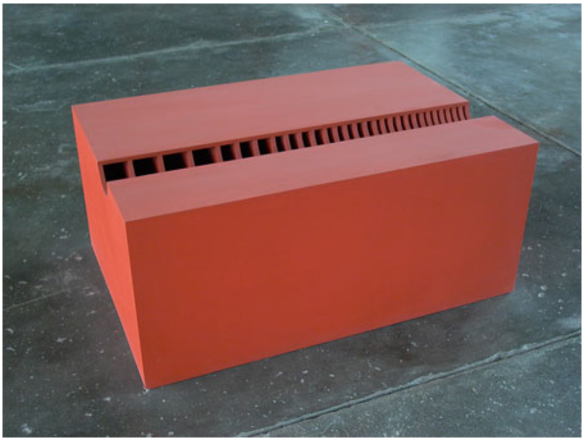

Fig. 3 Donald Judd, Untitled 1963 (DSS 41), 49.5 × 114.3 × 77.5 cm, Judd Foundation Marfa, Texas. Licensed by VAGA, NYC/Bildrecht Wien 2014. Courtesy of Judd Foundation Archives. Photo ©Judd Foundation

design of volumes according to proportions, a certain firm numeric relationship must be used [3, p. 123ff].

The title of the group of works known as *Progressions* corresponds in a meaningful way to the English expression 'series of numbers' or 'number sequence'. Judd resorted to special mathematical series, such as the arithmetic, geometric or alternating harmonic progression, as well as to the Fibonacci series, which he had found inter alia in the 1961 second edition of the book *Summation of Series*.[4] But while Judd created his schemes from number series of mathematics, he then went on to transform them. For example, in *Untitled 1965* (DSS 60) he crossed over parts of the infinite Fibonacci sequence 1, 2, 3, 5, 8, 13, ... in the opposite direction by choosing the first five parts for the regularly increasing right-to-left lengths of the orange-red bars (positive volumes), and selecting the first four parts for the increasing left-to-right lengths of the gaps (negative volumes). Thus, the complete scheme of *Untitled* 1965 (DSS 60) is 8, 1, 5, 2, 3, 3, 2, 5, 1. The reconstruction of the original mathematical number series—from the scheme with which the work generated, to a sign for the scheme, and to a kind of code for the number sequence—was particularly tricky. In the worst case even mathematical facts could be associated. In contrast to Mario Merz, Judd wasn't interested in the Fibonacci sequence as a metaphor for the natural and growth. Obviously Judd saw no need to set up the schemes and works of art so that the original number sequences in mathematics could not be reconstructed. On the one hand, this could have occurred through a suitable translation of the number series into schemes which would made a reconstruction possible only with difficulty, and on the other hand through an appropriate translation of the schemes into formal qualities of the works, which

[4] Robert Smithson had refered to this manual of mathemactics located in the library of Judd as one of the sources for the *Progressions*; see [36, p. 37]. In this manual alternating harmonic series (72) '$1 - 1/2 + 1/3 - 1/4 + 1/5 - \ldots \infty$' is indicated, which was used in *Untitled* 1965 (DSS 84); see [19, p. 14].

doesn't permit a deciphering of the schemes. While in *Untitled* 1963 (DSS 41) it is almost impossible to reconstruct the original number series because of the large number of parts of the scheme and the delicate changes in the distances of the lamellae, this turns out to be easier in the *Progressions*, such as *Untitled* 1965 (DSS 60), created 2 years later, because of the small number of elements and the integers of the number sequence. If Judd omitted to take precautions, even though the factuality of the available material was important to him, this must be explained by his statement: 'No one other than a mathematician is going to know what the series really is. You don't walk up to it and understand how it is working, but I think you do understand that there is a scheme there ... ' [9, p. 47ff]. This leads to the assumption that Judd modified the mathematical number series essentially according to formal and aesthetic aspects, while he mostly selected only a small number of parts, beginning with the first summand or subsequent part in the order of their appearance as figures.

The possibility of identifying the schemes in connection with Judd's examination of the question of the visual could lead to the erroneous conclusion that he wanted to visualize the mathematical through his art. This aspect was proposed by Mel Bochner in 1966 for the artists of the exhibition *Primary Structures* (even though Judd didn't present any *Progression* there): 'Old Art attempted to make the non-visible (energy, feelings) visual (marks). The New Art is attempting to make the non-visual (mathematics) visible (concrete)' [8, p. 34]. The fact that Sol LeWitt, although he himself experimented since 1963 with numerical orders in his works of art, as in *Wall Structure* (1963), considered Judd's *Progressions*, which he has seen on the occasion of the exhibition Art in Process at Finch College, as an important impulse for his work with the mathematical in the *Serial Project #1*, indicates that LeWitt valued the better realized visualization of the mathematical in the *Progressions* in contrast to his own works of art [10]. Because of this, the relation he found interesting between the empirical and the intelligible could be established, and he pursued this in his *Serial Project #1*.

Mathematics, Structuralism and 'Structure-Seeing'

To understand how the identification of the schemes by a contemporary observer might work (without having any specialized knowledge of mathematics), it is informative to recall the structuralism that was fashionable in the 1960s. This is also applied to mathematics, at a time when so-called structural mathematics was 'in'. Thanks to the New Math, the New York artists were probably as familiar as the average American with the set theory-based and structure-oriented contemporary mathematics.[5] Both the ethnological structuralism represented by Claude Lévi-

[5] Kynaston McShine mentioned in the catalogue *Primary Structures*: 'Contemporary art, inevitably, will be influenced by new intellectual concepts, and it is not surprising that many of the sculptors [Sol

Strauss and the linguistic structuralism of Roman Jakobson, the two working in New York, were inspired by mathematics, but can only be mentioned here. For our purposes, the key point is that structuralism had an effect on the theorizing and the practice of seeing, as verified by the activities of the artist and scientist György Kepes and the art theorist Rudolf Arnheim.

Being under the influence of the structuralism, Kepes and Arnheim, both of whom were also working in New York, represented a mode of seeing that can be described as 'structure-seeing'. Already in the 1950s György Kepes had noted a change from 'thing-seeing' [21, p. 204ff.] to 'pattern-seeing', which he saw as connected to the change from an atomistic to a structural viewpoint in the sciences. While in thing-seeing subjects would be picked out from the empirical data of the sense (*Sinnesdaten*) of perception, static entities would be produced and qualities of these subjects, such as their size, their form, their color and their substance would be perceived. In contrast, pattern-seeing overcomes the approach of focussing on and isolating the single object in favour of a relational, integrative seeing which contains a comprehension of phenomena in light of the dynamic interactions between objects [21, p. 204 ff.]. In accordance with the paradigm of structuralism, the comprehension of an object arises from the comparison with other objects and through the analysis of its position within the mutual relations [15, p. 18]. To Kepes structures constituted a link between the arts and sciences, because scientific recognition and artistic seeing cannot be separated from each other: 'Each historical era seeks and needs a central model of understanding. Structure seems central to our time—the unique substance of our vision' [22, p. ii]. That Rudolf Arnheim was also dealing with something like structure-seeing can be discerned in his characterization of 'active seeing', in his 1954 book *Art and Visual Perception*: '... vision is not a mechanical recording of elements but the grasping of significant structural patterns' [4: p. viii]. Structure-seeing was shown to its fullest advantage in his book *Visual Thinking* (1969), where he discussed the didactic advantages and the operation of visual media of mathematics and their role in the visual thinking of his readers. Using the example of a mathematical bar chart, he demonstrated the practical application of structure-seeing (Fig. 4).

Structure-seeing can actually be tested and trained ideally in mathematical diagrams which represent structures of mathematics as 'structure pictures' (Strukturbild) [34: p. 201ff]. While on the left side of the number line in the middle, which serves as a scale, vertical bars can be seen in descending height for the corresponding numbers 1 to 10, the four configurations on the right side depict possibilites of the summation of the number 10 in an illustrative (anschaulich) way [5: p. 219ff]. To Arnheim this manner of graphic representation offered the possibility of 'the grasp of perceptual structure' [5: p. 212], which is to be preferred to 'counting or measuring'.

LeWitt, Donald Judd, Robert Smithson, Carl Andre and others] have found the new physics and the new mathematics as a source of ideas which open new possibilities for them' [30].

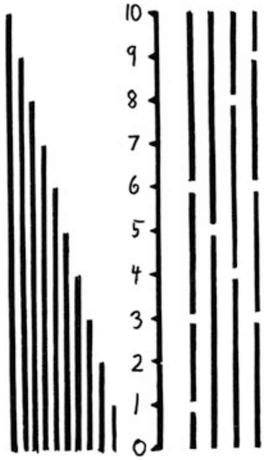

Fig. 4 Sum-diagram in Rudolf Arnheims *Visual Thinking* (1969) ©2004 by the Regents of the University of California. Published by the University of California Press. Image: [5, p. 219]

How Knowledge (of Visual Culture) of Mathematics Produces the Mathematical

Evidently, knowing about the visual culture of the mathematics, such as dealing with the bar chart just introduced, the observer of Judd's *Progressions* was familiar with the translation of line elements ordered rhythmically by looking at to numeric values. The analogy of the processes becomes even clearer, if the bar chart is turned clockwise by 90° and the formal similarity of the line configurations to the *Progressions* becomes obvious. Structure-seeing supported the comparison of the elements which was necessary as in the bar chart. The art historian Michael Baxandall proved that both the production and the reception of works of art have to be understood in connection with the knowledge of the artists and the observers of contemporary culture, particularly of visual culture,[6] which explicitly includes the knowledge of mathematics and its visual culture.[7]

Baxandall described the procedure of seeing in the perception of art as ‘period eye’ [6: p. 29], which is dependent on culture and tied to the time and the place of consideration. Along with the concept of the ‘period eye’, two other prominent mathematical media are indicated—the number line and the Cuisenaire rods, which normally serve as aids in conveying mathematical facts—which, due to the formal similarity with the *Progressions*, could connote the works as mathematical, while at the same time especially enabling the perception of the existence and the identification of number-values because of the ability they fostered in practical

6 The idea of the visual culture shall be regarded here as the totality of visual media of a community or required for a discipline and the related experience; see [35: p. 9].

7 In contrast to the Renaissance artists mentioned by Baxandall, New York artists like Judd didn't aim at the mathematical pre-knowledge of their observers in the context of an intellectually stimulated game. The concept is nevertheless applicable.

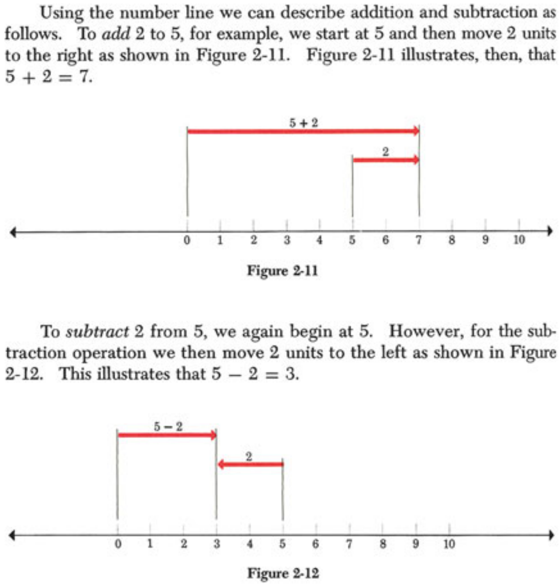
Using the number line we can describe addition and subtraction as follows. To *add* 2 to 5, for example, we start at 5 and then move 2 units to the right as shown in Figure 2-11. Figure 2-11 illustrates, then, that $5 + 2 = 7$.

Figure 2-11

To *subtract* 2 from 5, we again begin at 5. However, for the subtraction operation we then move 2 units to the left as shown in Figure 2-12. This illustrates that $5 - 2 = 3$.

Figure 2-12

Fig. 5 Exercises of addition and subtraction of lengths using number lines. Image: [14, p. 92]. Image: used by permission of the estate of Roy Dubisch

dealing. The work with the number line also enabled the observer to recognize the length of intervals by seeing and denoting them with numeric values. In American mathematics books of the 1960s, such as *Exploring Modern Mathematics*, the measurement of lines as well as the determination of relative lengths was given either by visual comparison with a predefined unit or by using a ruler with a scale [20: p. 293ff]. In the book *Mathematics I* [14] the addition and subtraction of integer numbers was illustrated by joining together lines with the help of suitable diagrams [14: p. 92ff, p. 421ff] (Fig. 5). The number line equated with a ruler and the red bars put on top of each other resemble Judd's *Progressions*. The sticks named after their inventor, the Belgian primary schoolteacher Georges Cuisenaire (1891–1976), are a set of small rods of wood, square in section and of various lengths, and colour-coded by length, in which the white colored cube depicts the unit size. Within New Math they were popular auxiliaries for mathematics teaching since the 1950s.[8] The sticks served to illustrate the treatment of simple arithmetical facts such as the addition and multiplication of numbers, as well as more complex operations. The example taken from *Life Magazine* 1961 demonstrates the configuration of sticks to illustrates the addition of $2 + 2 + 2 = 6$ or $1/3 + 1/3 + 1/3 = 1$, or the multiplication $3 \times 2 = 6$ [28: p. 47] (Fig. 6).

[8] Linda Dalrymple-Henderson suggested the consideration of the Cuisenaire rods.

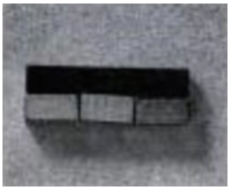

Fig. 6 Cuisenaire rods. Detail from *Captivating. Key to Math*. Image: [28, p. 47]. The right holder could not be found out. To whom it may concern please contact us

Like in Judd's 'square-nosed' [9: p. 49] *Progressions*, the cuboid-shaped constituents are joined together additively and assembled in an upper and a lower line. In contrast to the bar chart, the ability to determine the numeric length of plastic objects was explicitly exercised with the Cuisenaire rods. As in Judd's *Progressions*, it occured by comparing the lengths of different but uniform objects, particularly with the unit of the rod of length one. Similar to Michael Baxandall, the assumption could be expressed that Judd had been inspired to his *Progressions* by the Cuisenaire rods, which were surely well-known in artistic circles.[9] In particular, Robert Smithson wrote: 'Judd has a labyrinthine collection of "printed matter," some of which he "looks" at rather than reads. By this means he might take a math equation, and by sight, translate it into a metal progression of structured intervals' [37: p. 18]. Remarkably, the construction of the *Progressions* was responsible for the particularly good recognisability of the embodied schemes as the arithmetical basis of their rhythmic.[10] For this reason they were predestined to award the connotation of the mathematical in Judd's oeuvre. To extract the original number sequences out of the schemes, it wasn't necessary to be a mathematician or to know mathematics books. The Fibonacci sequence is part of the cultural memory and was present in Modern art, and in contemporary art as well, through the work of Mario Merz.

The Combinatorial Rule and the Visual in Sol LeWitt

Clearly speaking, Sol LeWitt believed his work of art to be understood if the observer had understood the principle of its construction, or as he once said: 'Once the process is discovered, the piece is understood'.[11] This refers to the systematics of the changes of the 36 configurations in *Serial Project #1*, which is frequently

[9] After their introduction in the United States in the 1950s, the Cuisenaire rods were reviewed in many popular journals and magazines such as *New Scientist*, *Scientific American* and *Life Magazine*, as well as in the first edition of *Whole Earth Catalog* (1968), which was very popular among artists.

[10] Furthermore, the mathematical content of the *Progressions* was reviewed in publications and explanatory material presented at exhibitions.

[11] This statement of Sol LeWitt refered to the concept and the realization of his work *Incomplete Open Cubes* (1974). See interview of Russell Bowman with Sol LeWitt, VHS-Video, Museum of Contemporary Art Chicago 1979. Transcription by the author.

described as a 'premise' [27], 'system' [29: p. 43], 'program' [27] or other, similar terms by LeWitt himself and in the literature. Accordingly, the artist explained in an accompanying *Cahier*: 'The premise governing this series is: to place one form within another and include all major variations in two and three dimensions' [27]. The configurations are the result of the two parameters, 'finish of surface' (open/closed) and 'height' (low/middle/high) of the two cuboid forms nested into each other, and the 36 possibilities completely tried out.[12] It is appropriate to talk about a rule that is combinatoric, formal: on one hand, the rule concept points out an action according to a 'regulation' [26], namely of the production of the work of art on the basis of the rule. On the other hand, the rule concept expresses the generative potential of LeWitt's premise according to an algorithm, since probability cases are laid out. Understanding combinatorics like the mathematician George Pólya did, as 'the study of counting various combinations or configurations' [33: p. 2], LeWitt's premise has a combinatorial character. Using a technical vocabulary LeWitt talked about 'possible permutation' [27] and 'finite series' in his *Cahier*. Since the possible configurations in the combinatorics by the term $2 \times 2 \times 3 \times 3 = 36$ is portrayable and calculable, the rule is to be formalized according to Sybille Krämer.[13] Therefore the mathematical is a formal, combinatorial rule in the *Serial Project #1.* In contrast to Judd, it was essential to LeWitt's art that the observer is able to discern the mathematical in his works. For this reason, in his *Serial Project #1* LeWitt relied less on mathematical pre-knowledge than on structure-seeing. Knowledge of the visual culture, such as that of the bar chart with its height graduations, could be helpful in recognising the height variations in the *Serial Project #1* and in arriving at the rule with a combinatorial trained intellect mediated by the New Math. Since the formal differences of the configurations are strictly deduced from the combinatorial rule, the rule could be reconstructed from the differences by a comparative structure-seeing directed towards relations. It is hardly surprising that in 1967 in his Cahier LeWitt discussed the 'differences between the parts', a year after a reprint of Ferdinand de Saussure's 1916 *Course in General Linguistics* (*Cours de linguistique générale*) had been published in New York [11]. Here the father of the linguistic structuralism explained that signs received their meaning not by the reference to something absent but by their difference to other signs within a system [12: p. 143ff]. Undoubtedly LeWitt's *Serial Project #1* is laid out in a structuralist version [31: p. 115f]. Comprehending the *Serial Project #1* as a script, the paradox mentioned by Lucy Lippard can be resolved, because then the stereometric volumes correspond to the letters as significants of a language [7: p. 42f]. If it is the differences between the cuboids and the concrete forms which are meaningful, then it is understandable that LeWitt could say, that in principle the appearance of a work of art doesn't

[12] A single form can take three steps of height and be respectively open and closed. This produces respectively $3 \text{ x } 2 = 6$ possibilities. The arbitrary combination of the conditions of inner and outer form generates $6 \text{ x } 6 = 36$ configurations.

[13] This can be depicted 'with the help of artificial symbols', and this depiction fulfills the qualities of the 'typographical, schematic and use of symbols without interpretation' [23: p. 1f].

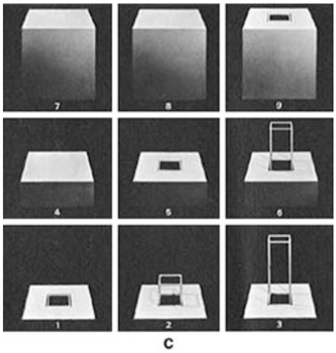

Fig. 7 Sol LeWitt, photographic plate of set C of the *Serial Project # 1 (ABCD)*. Licensed by VAGA, NYC/Bildrecht Wien 2014. Image: [27]. The right holder could not be found out. To whom it may concern please contact us

matter. In this light the design of the single configuration as well as the colour is important for determining the rule only when it shows suitable differences.[14] The differences only made sense because of their connection through the combinatorial rule and their feedback. Decisively, the work of art contains an intelligible part from the moment the combinatorial rule is recognized, which makes it at the same time self-referential and idealistic. Idealistic, because the material work of art becomes a manifestation of the rule of an ideational entity and self-referential, because the rule points out the construction of the work of art. Operator and operandum intersect each other through the rule. Moreover, knowledge of the rule has an effect on the visual perception of the work of art in a kind of feedback effect. Since the observer is informed about the construction of the work, he can ‘see’ what actually can’t be seen. Concealed or invisible elements, such as the inside of the configuration 8 of the set C could be imagined (Fig. 7). A criticism of the neutrality of the geometric forms of Minimal Art expressed by the art critic Michael Fried in his famous essay *Art and Objecthood* of 1967 can be connected to that. Fried stated that the objects of Minimal Art are ‘blatantly anthropomorphic’ and ‘disquieting’ because of their ‘hollowness’ [16, p. 16ff]. On this basis Georges Didi-Huberman created in the 1990s a picture conception whose visuality was provided with a ‘negative index’ [24: p. 12]: It is not the visible surfaces but the non-visible parts of a work of art like a picture or a sculpture which could beset us. The hidden and the cavity of a volume would especially create a subject-like object which can frighten the observer, as Didi-Huberman explained [13: p. 63ff]. If LeWitt demystified the invisible inside with the rule without having to make the form of it comprehensible, then he could counter the criticism and weaken it. The relationship of the empirical and the intelligible of *Serial Project #1* was contemporaneously connected with the criticism of the comprehension of seeing as it circulated in the discourse of visuality. According to Arnheim’s active ‘seeing’, pervaded by thinking, LeWitt pointed out the importance of thinking for seeing and therefore for the perception of art.

[14] It is worth noting that years earlier Sol LeWitt already started to paint the parts of the series Structures white to neglect the materiality which was unimportant to him. In contrast, the research field *Schriftbildlichkeit* focuses on the aesthetic dimension of documents; see [25].

The interpretation of a work of art, as *Serial Project #1* has taught us, is not produced directly by its phenomenality but indirectly through the discernment of an inherent, intelligible rule interacting reflexively with the visual.

The comparison of how Sol LeWitt and Donald Judd refer to mathematics and use the 'mathematical' reveals not only a historical development of New York art, but also shows varied intentions and thus the individuality of artistic positions of so-called Minimalists. The 'mathematical' in the form of a combinatorial rule, is an essential component used by LeWitt to analyse how art is seen and perceived; with reference to Judd's work, the concept of production could interpreted as a phraseology of arithmetic for reckoning with Europe. Similar to Michael Baxandall's concept of the 'period eye', knowledge of the visual culture of mathematics enabled the observer to recognize the mathematical in the works. This process was supported by the concept of 'structure-seeing' within the theory of perception, characterized by mathematics and structuralism. To Judd, problems of formality and aesthetics of effects such as *wholeness* were important in his recourse to numeric 'schemes'; LeWitt, however, puts the formal aspects of a work of art, the formalism into question.

Translated from the German by Bettina Marten.

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In-tensioni reciproche: Performance

Emanuela Fiorelli, Caterina (Katia) Di Rienzo, Massimo Cappellani

Visual Tension

The performance entitled *In-tensioni reciproche* (Reciprocal In-tensions) was born from the idea of tension, or better, of the power of tension, a power that is always present even when invisible to the eye.

It is instead visible in its consequences: extreme resistance, a clean break.

In the installation created for the performance I created an elastic surface anchored to the wall.

The surface was discontinuous, instable, active, untiringly taut, sensitive to any small movement, never at rest. It reacted to all solicitations with the same force that was exerted on it. It allowed itself to be deformed, perforated, lacerated, and pulled, but once the action was finished, it returned to its place.

The elastic thread is a pencil mark that takes off from the surface of the page to seek anchorages and extend itself into space, connecting surfaces and environments. The relationship is intentional, but is also forced. While my intention was 'a moving towards', the nature of an elastic thread is 'a returning to'.

This dualism is such that it creates an apparent immobility, apparent because it is the result of the tension created between the elastic threads that want to return to a position of rest and the wall created to provide resistance.

In my works, tension is the condition for beauty; if a thread breaks, or unravels, the highest quality of this material—the elasticity—decays.

Every material has characteristics that distinguish it from all others, and it must be used for those characteristics.

Michelangelo exalted the nature of marble because in the famous 'unfinished sculptures' he stopped taming it and extracted from the block, not the image of an idea, but rather the very nature of the marble.

Emanuela Fiorelli (visual artist), Caterina(Katia) Di Rienzo (choreographer, philosopher), Massimo Cappellani (visual artist)
Rome, Italy
https://www.youtube.com/watch?v=9DWTpyrmByM
Galleria Verrengia, Salerno, May 2012; Galleria Fabbri C.A., March 2013.

Emmer M. (Ed.): Imagine Math 3. Between Culture and Mathematics
DOI 10.1007/978-3-319-01231-5_ 9, © Springer International Publishing Switzerland 2015

Photo 1 *In-tensioni reciproche*

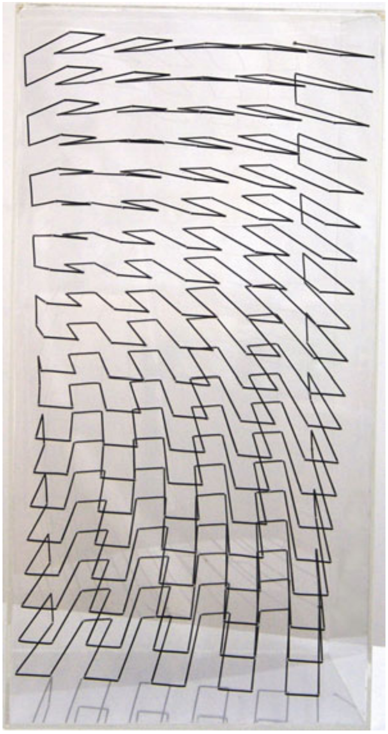

Photo 2 E. Fiorelli, ST: $24 \times 24 \times 48$, 2008

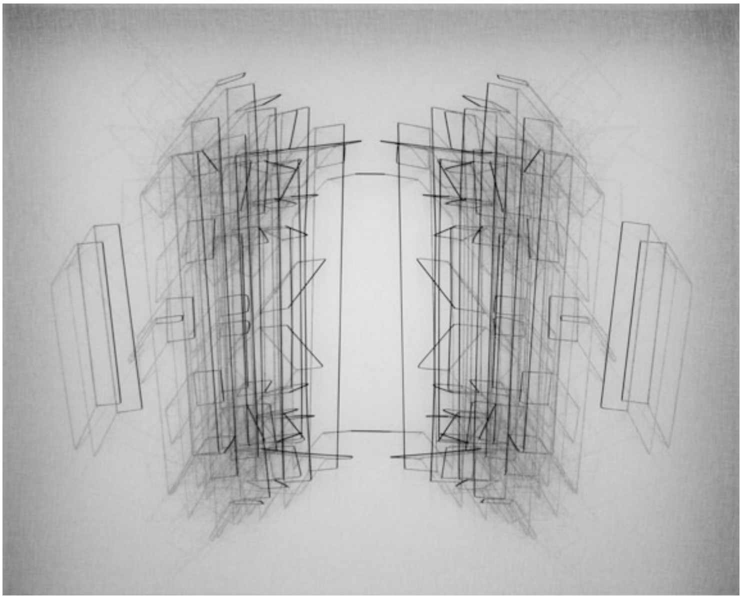

Photo 3 E. Fiorelli, *Yantra*, 103 × 127 × 10 cm., 2010

Photo 4 E. Fiorelli, *10 metri*, 50 × 50 × 25 cm., 2013

Photo 5 E. Fiorelli, *Enunciazione*, 2011

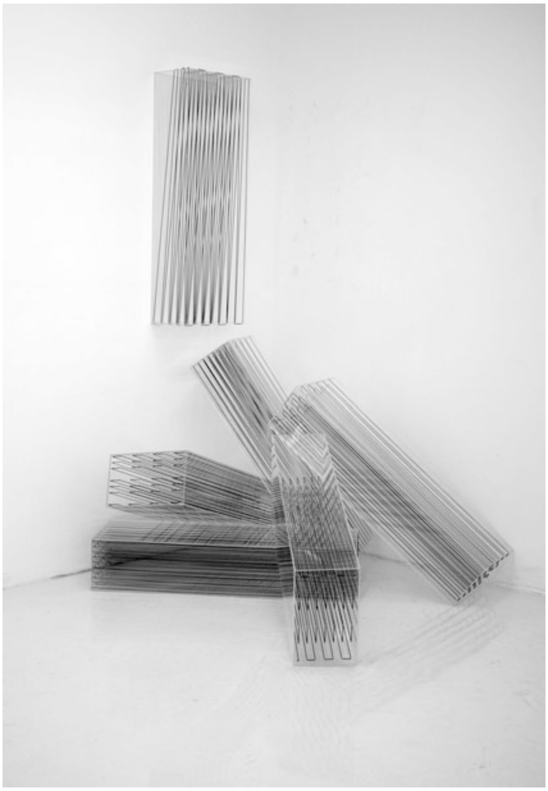

Photo 6 E. Fiorelli, *Le macerie del trascendente*, 6 boxes, each 100 × 35 × 20 cm., 2012

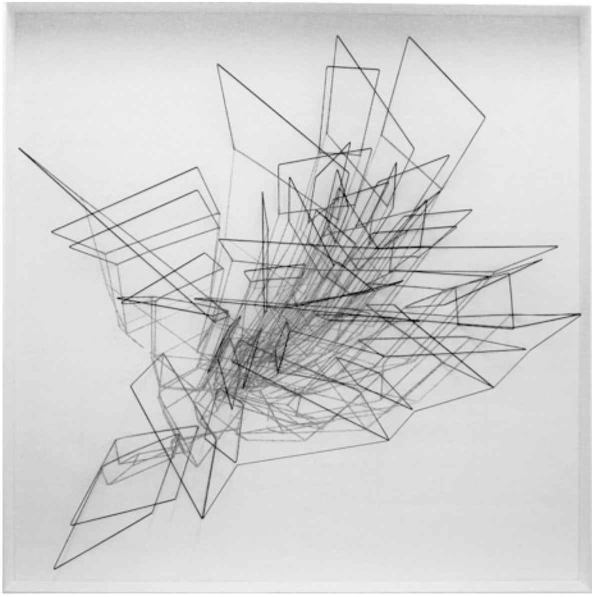

Photo 7 E. Fiorelli, *Crescita*, 116 × 116 × 11 cm., 2012

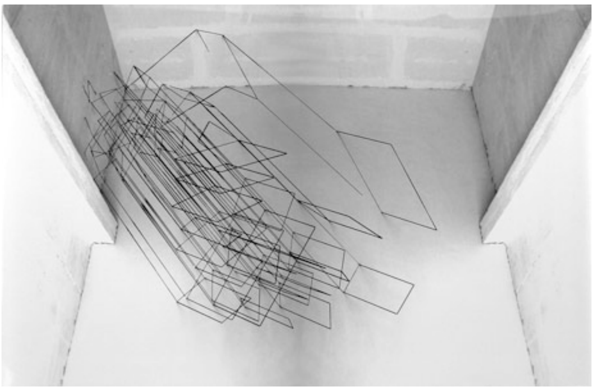

Photo 8 E. Fiorelli, *Aeroscultura 1*, 72 × 111 × 10.5 cm., 2013

Elasticity is the invisible force that holds the treads in tension, and in my works this relationships created harmonic designs, aerial views, configurations that perhaps structure our very existence. Some of the designs are regular, others emerge from chaos to hypothesise possible laws. In some the repetition of chaos confirms

the rule; in others the change in point of view obscures and complicates the perfect legibility.

Many works are born with a knot and finish with a knot, uninterruptedly: a single thread that produces pathways between the beginning and the end.

We are able to see the aerial view of our design only at the end; this might be a metaphor for life. A design can be a layout of lines on a plane surface, or the plan of something else which, to clarify itself, needs to be extruded in space.

In fact, the thread is a three-dimensional mark, and with it I create, between the surfaces, a modular weave that takes the place of a flat, artificial view of space on paper, a view that is real and measurable. Some works contain two designs that are mirror-images of each other; could this be because the eye seeks an equilibrium between opposites? Or perhaps because the focal point is not in the design, but tends towards 'beyond' (and thus the foreground is automatically duplicated)?

As Francis Ponge writes in *L'homme à grands traits*:

> symmetry can be the effect of vibrations. … Make a cord vibrate and you will see this. Thus two eyes, two arms, two lungs, two kidneys, two legs, two hands, two feet. But when the vibrations diminish … they all tend two one … [1; my trans.].

In the works of the last 3 years I have also integrated the elastic threads with photographic images in order to add ambiguity to our understanding of what we see.

I personally photograph the environments that make up the backgrounds for my threads, but what turns out to be seen by our eyes is not the thread with the background behind, but rather a genuine installation that opens before us and can be entered. However, it only has a depth of 10 cm.

Imagining Dance

> … we shall carry our thought a little further: a little further beyond the customary, immediate idea of the dance [2, p. 64].

Dance is an art that departs from the visibility of objects and enters that of bodies, which are the material it uses, a material that is, all things considered, ordinary—we dance with the same body we live with. Nevertheless, it makes extraordinary use of this ordinary material: human gestures can 'say' what has not yet been felt, compose that which has not yet been seen. Like an immense transparent body, this incidental flesh causes 'invisible images to exist' [3].

This is not a metaphysical perspective. Dance is an art of our finiteness, whose limits are in any case stretched by it. It deconstructs exterior visibility, revealing unknown figures. This is because it is a real experience that is provided through an imaginary one. The dancer is a body that is capable of containing in the physical world, other worlds; not only real bodies, but possible bodies.

Imagination is the true, great power of dance; its tendency is to create, not an imitation of life, but rather its transfiguration. In the choreographic event, the dancer

Photo 9 C. Di Rienzo, *Fogli*, 2011

and the spectator, the one who acts and the one who enjoys, are part of an unusual perception: the existence of man and the situation of the world recreated in the imagination.

Dance, we might say, is an imagining transcendence of the visible to the invisible, where we think of the imagination in Michel Foucault's terms [4], according to which imagining is not an act of a consciousness that flees the world and aims for the quasi-reality of the image, but is a universe to be unfolded within the heart of the existent, thanks to the power to transcend it, imagining inside it while we perceive it, rather than imagining over it in its absence.

It is interesting to consider a situation of imagination such as this in light of the *utopique* described by Michel Bernard. This is a completely new idea of utopia, identified in the capacity of art, and above all that of dance, to make an icon of another aspect of the world, from within it. That is, utopia is no longer a socio-institutional model that is hoped for and in the future, to be constructed outside ourselves. According to Bernard, utopia is declined in *utopique*, that is, in that particular tonality that an action or gaze assumes when they succeed in subverting the constitutive vision of reality, when they become deconstructive perception, when, that is, they involve the power of imagination. True *utopique* power is that which 'inhabits and animates dance', 'immanent in the movement itself of dancing' [5, p. 24; my trans.]

To imagine dance is to see another existence, elsewhere and yet there, in the world that dances. It is to remain inside forms that re-signify reality, in a great significance that grasps the human condition, evoking invisible images. To imagine dance is to see in the body how significance exceeds the signified, how the way of signifying exceeds the thing signified.

Photo 10 C. Di Rienzo, *Fogli*, 2011

Dance, we might say, represents a sort of nascent state [6] of the image, the complex movement from which image draws its origins. If from the point of view of a genealogy of image it is possible to affirm with George Didi-Huberman that "every image is the result of movements that are temporarily sedimented or crystallised within it" [7, p. 39; my trans.], from the point of view of the rules of dance, in a way that is symmetrical and opposite, we can say that dance is an image where movement is not arrested, it is image-body, image of the flesh. In dance the body becomes image and image is living flesh.

Perhaps such consideration lies at the base of one of the most innovative trends on the international dance scene. The recent intersections of contemporary choreography in fact display a front of research in which dance meets visual art, not only to establish a dialogue between different arts, but to create a new and different art form.

This is the poetics that sustains *In-tensioni reciproche* where dance, at the crossroads of installation art and photography, expresses its internal vocation to be encounter and transit. Gesture broadens to the visual, and extends to the architectonics of the installation. In this synthesis of performatory and visual, the three elements of space become planes of reciprocal crossings, which rewrite the visible.

In the performance this creates a new aesthetic equilibrium. The body is not alone, but is interwoven with other signifying forms, elastic threads and projected photographic images. These three visual elements generate something we might call 'effects of surface'.

These function as an exploration of the structures of the surface. Like layers of film, they stretch and extend stratifications of visibility; they are somewhat similar to

Photo 11 C. Di Rienzo, *Fogli*, 2011

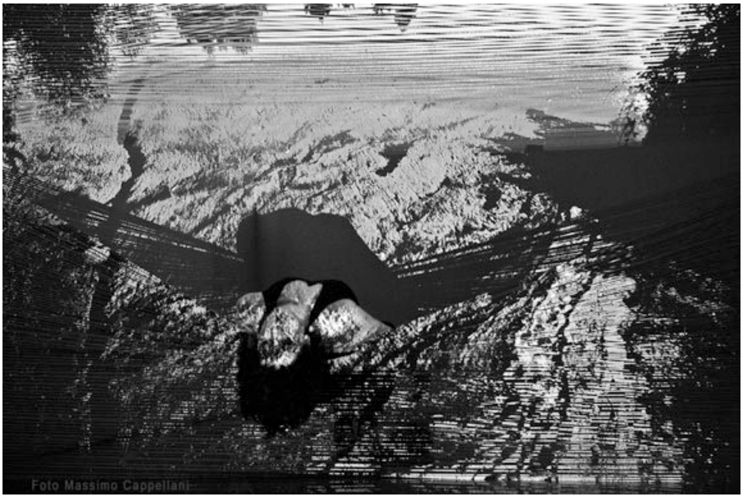

Photo 12 C. Di Rienzo, *In-tensioni reciproche*, 2012

what Mario Perniola conceives in terms of a rehabilitation of the concept of depth, based on the idea that depth is not an inwardness, but in fact a stratification, where the focus is on the experience of outwardness [8], but one which deconstructs the superficial in favour of a surface that is stratified, dense.

Photo 13 C. Di Rienzo, *In-tensioni reciproche*, 2012

Investigating the relationship between dance and cinema, another relevant intersection in contemporary dance, Norbert Corsino has spoken of the body as *l'écran tatoué*, a 'tattooed screen' [9].

The skin is a kind of screen, an iconic 'membrane' where dance takes place. It is visual quality that is found in our work due to both the dimension of the projected image as a 'moving tattoo' of the body, and to the lines, marked by the elastic threads, like a pathway of significance on which the body engraves discontinuities, making openings like on a screen.

Photo 14 C. Di Rienzo, *In-tensioni reciproche*, 2012

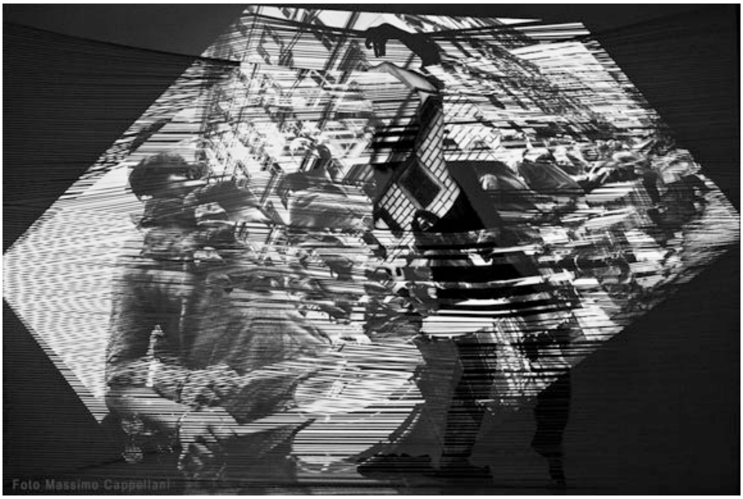

Photo 15 C. Di Rienzo, *In-tensioni reciproche*, 2012

Through the concept of tension, this research, we might say, is further declined, because the three surfaces, *in reciprocal tension*, enter into a tension that is structural, symbolic, imaginative, not only physical. It is a tension that we may say evokes the *tonos* of the Stoics, that is, tension as a consonance, cohesion, reciprocity. It is tension as the search for intersections between arts that generate expression.

The Performatory Photography

> Thus it falls to the photographers, the archaeologists of surface, to follow, with their cameras, the traces of the silent objects and to extract from the infinite abundance a piece of existence, a fragment that, detached from disorder, becomes new, pure, never before seen [10, p. 64; my trans.].

With regard to the theoretical perspective of Georges Didi-Huberman, it is important to me to highlight what might be his poetic development, that is, the possibility for the photographer to utilise the same fragments extracted from the disorder of reality to construct a narrative whose content is 'new', also with respect to the novelty of the individual pieces of existence. To construct, substantially, 'content to the second power'.

Before speaking of the current outcome of this possibility in the work *In-tensioni reciproche*, it is necessary to reconstruct, briefly, the conceptual process that led to the definition of content to the second power, a compositive hypothesis that gave rise to the proposal of a performatory photography.

At the basis of the question lies the tension produced by the binomial form-content. This is a question which, from my point of view, finds a synthesis in analogue photography in black and white, a technical-formal choice that is also a choice of content, in the sense that it is a photography in analogy with the world whose essence it seeks, an 'archetypical photography' [11], a privileged route to grasping those aspects concealed by reality, Didi-Huberman's 'fragments'.

In *A History of Six ideas*, Wladyslaw Tatarkiewicz, analysing the concept of form, speaks of its being raised to a power; more precisely, he speaks of a 'form to the second power' [12, p. 221]. In this expression he combines two different meanings: form as composition and form as what is presented to the senses, that is, as that which 'denotes the exterior aspect of things'. Thinking of an image, therefore, form to the second power is constituted of both the arrangement of the 'objects' in the visual scene, as well as by the 'aspectual' elements of the aspects of those same objects. We can therefore state that this definition is valid for every single image, as a 'product' of two different terms.

Still remaining in the discourse of the Polish philosopher, it appears possible to identify content through the same two accepted meanings of form, as complementary to them. With respect to the form as 'composition', the content concerns 'the elements, components, or parts which form unites or melds into a whole' [1980, p. 220]; with respect to form as that denoting the exterior aspect of things, we might

say that the content is expressed by the meaning, by the intrinsic significance of ‘things’.

The content to the second power to which I have referred does not regard the content of an individual image, nor that constructed by an entire photographic story; it goes beyond the narrative of a specific place or a particular event, that is, beyond the canons of spatial coherence and temporal duration. Instead of the interpretation of a place or a fact, the perspective I am referring to proposes, rather, to express a meta-story, or to arrive at constructing an ideo-logical discourse. To do this, I avail myself of images that regard diverse places in diverse times, of fragments that do not have any space-time homogeneity but are united by an interior weft, which is precisely content to the second power.

The project *Danza-solo per fotografia* (Dance-only for photography)—developed from 2008 to 2010—is the first work in this direction; in fact it seeks to construct an ‘inversion’ of dance and photography, of visual and performance. Gesture becomes image, the photography becomes performance, and in this reversing the dimensions of time and space lose their centrality, motion and stillness come forward.

Photo 16 M. Cappellani, *Danza-solo per fotografia—Vulcano*, 2009

A project like this allowed me to intuit that by further broadening the field of interactions it would be possible to achieve an ‘exponential’ expansion of the ‘contents to the second power’. This occurs through the interaction of diverse artistic languages.

This brings us to *Fogli* (2010–2011), a work created around a philosophical theme, together with Caterina (Katia) Di Rienzo for choreography and dance, Roberto Biondi for architectonics and Vincenzo Pennacchi for painting and

Photo 17 *Fogli*, 2011

sculpture, and to *In-tensioni reciproche*, an experimentation that began in 2012 and carried on in 2013, which represents the current results of this research.

The succession of images projected on the elastic threads and on the body move on the double register of form and content. From the ambiguous rigidity of moveable geometries, to the disorientation of the abstract sign, from the perceptive polarity of the image (positive–negative, softness-harshness), to the dynamic deformations of the material, the photographic video projection with its explicit or symbolic contents, through direct narrative or evocation, recounts its own vision of the world, playing on the subtraction and inversion of figure and ground, light and shadow, black and white, rights and oppressions.

Here we see that it no longer makes sense to consider the contents of an individual 'image'; everything has become a 'meta-image', contents to the second power,

Photo 18 *Fogli*, 2011

a logical weft; indeed, paraphrasing what Mario Dondero had to say about my photograph, an ideological weft.

To conclude, I would like to highlight the enormous extent to which the circular pathway given by reality–photography–video editing–projection–re-writing of reality, grafted into the crossroads between the diverse expressive languages, has broadened the contents to the second power, so much so that, in my opinion, rather than a simple re-writing of reality, we could speak of the construction of a hyper-reality, of a world that lies—or perhaps at this point we might say moves—in a dimension parallel to our own, close but unattainable.

Translated from the Italian by Kim Williams

Photo 19 *In-tensioni reciproche*, 2012

Photo 20 *In-tensioni reciproche*, 2012

Photo 21 *In-tensioni reciproche*, 2012

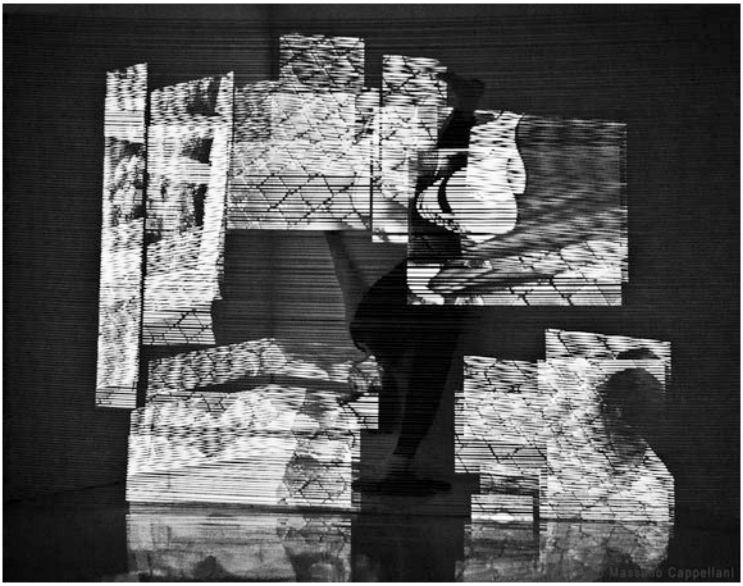

Photo 22 *In-tensioni reciproche*, 2012

Photo 23 M. Cappellani, *D.A.I.*, 2012. Complete image used in *In-tensioni reciproche*, 2012

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Part II
Tribute to Mandelbrot

The Fractal Organization of the Nervous System

Gabriele A. Losa

Introduction

The original conception of Galeno (Pergamo) (129–216 D.C.), confining the superior functions of human brain within three cerebral cells (spheres), has spanned several centuries up to the Renaissance period culminating with Leonardo da Vinci (1452–1519). A first outstanding breakthrough was accomplished by Andreas Vesalius who, in his famous work *De humani corporis fabrica* (1543), described the surface cerebral convolutions even tough failed to provide a reliable identification of peculiar morphological pattern. Relevant investigations were successively performed by Marcello Malpighi (1628–1694) who suggested the existence of a nervous fluid filling within cerebral glands, by Thomas Willis (1621–1675) who evidenced an arterial circuit by anastomosis of internal carotids and vertebral artery, and by Vicq d'Azyr (1746–1796) who revealed convolutions in unidentified areas of the brain external surface. Albrecht von Haller (1708–1777) underlined the secretive function of human brain by means of a nervous fluid. Franz Joseph Gall (1758–1828) and Johann Spurzheim (1776–1828) by examining the brain shape identified phrenologic maps with specific functions. Paul Broca (1824–1880) localized cerebral functions such as the langage arguing that "Nous parlons avec l'hemisphère gauche". Carl Wernicke (1848–1905) identified an area of the temporal lobe, whose damage may provoke the selective loss of the capacity of listening words. Back in the early twentieth century appeared the outstanding contributions of two coeval scientists: Camillo Golgi (1843–1926) [1] who postulated that ramified nerve fibers could support the "reticular theory", considering that the nervous system is a syncytial system which consists of nervous fibers forming an intricate diffuse network along which the nervous impulse may propagate. On the other side Santiago Ramón y Cajal: (1852–1934) [2] for which the relationship between nerve cellsdeveloped the "neuron theory" for which the relationship between nerve cells was not one

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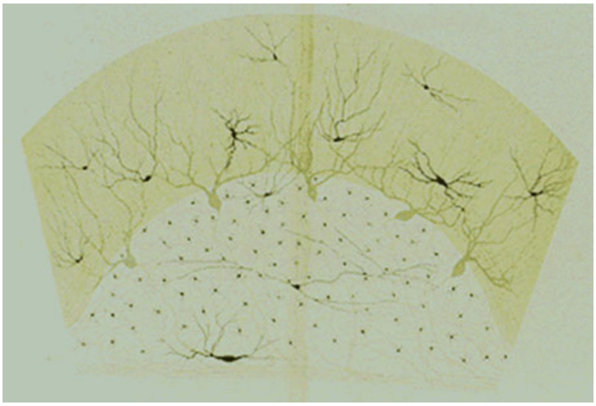

Fig. 1 The human cerebellar cortex as stained and drawn by Golgi (from the Opera Omnia, 1903). The staining procedure consisted on the reaction of silver nitrate with potassium dichromate that formed a black deposit within the soma, axon and dendrites of nerve cells of cerebellum

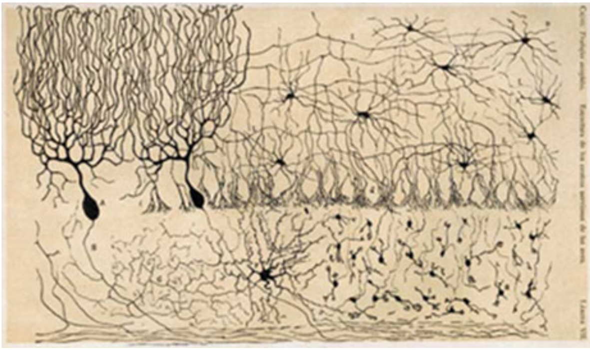

Fig. 2 The cerebellum cortex (Golgi staining) as drawn by Santiago Ramon y Cajal. (1888) Rev. Trim. Histol. Normal. Patol

of continuity, but rather of contiguity, accomplished through small membranous spines protruding from a neuron's dendrite that typically receives input from a single synapse of an axon (output) Piccolino [3] (Figs. 1 and 2).

In the last decades relevant new imaging techniques, such as Positron Emission Tomography (PET), Functional Magnetic Nuclear Resonance (fMNR), Computed Axial Tomography (CAT), etc., have been implemented that, in concomitance with the increase of the theoretical knowledge provided by the modern Mathematics and the innovative Fractal Geometry (power law scaling, self-similarity), have enabled to deepen into the morphological/ structural complexity leading to the analytical representation of the biological elements and to the objective description of living processes.

Main Properties of Fractal Elements

According to Mandelbrot [4] "A fractal set is a set in metric space for which the Hausdorff-Besicovitch dimension D is greater than the topological dimension DT." In nature, a fractal object is defined by its structural properties, namely by surface rugosity, irregularity and absence of smoothness, form invariance, geometrical or statistical self-similarity, morpho-functional complexity, represented by a fractional/non-integer dimension. The Richardson-Mandelbrot equation provides the mathematical basis for understanding geometric and spatial fractal structures, and for measuring and interpreting them, namely:

$$\mathrm{L}(\varepsilon) = \mathrm{N}(\varepsilon).(\varepsilon) \tag{1}$$

where $\mathrm{L}(\varepsilon)$ represents the contour length (e.g. the perimeter) of the biological component under investigation, (ε) the unit length of measure, and $\mathrm{N}(\varepsilon)$ the number of unit lengths (ε) needed to cover the contour $\mathrm{L}(\varepsilon)$. By substituting $\mathrm{N}(\varepsilon)$ with [lo D ε-D] into (1), the above equation can be transformed by logarithmic procedure and rewritten as:

$$\log\left[\mathrm{L}\,(\varepsilon)/\mathrm{lo}\right] = (1 - \mathrm{D})\log\left[\varepsilon/\mathrm{lo}\right] \tag{2}$$

which is the equation of a straight line with slope 1-D, and from which the dimensional exponent D can be calculated to yield the numerical value of the fractal dimension FD. FD is a statistical non-integer quantity that gives a measure of geometric complexity of form. lo is a reference scale without influence on the determination of D. In contrast to mathematically generated fractals, biological structures and objects observed in Nature are self-similar within a limited range of scales. Only within this scale interval or scaling window can the scale-invariant (fractal) properties of an irregular object of finite size be observed [5]. A real "fractality" exists only when the experimental scaling range covers at least two orders of magnitude, although fractality over many orders of magnitude has been observed in various natural fields. The fractal window characterizing biological and natural fractals also called "biasymptotic fractals" is graphically represented by the region II in the middle of three typical regions, limited by a lower ($\varepsilon_{\mathrm{min}}$) and an upper bound ($\varepsilon_{\mathrm{max}}$) of the bi-asymptotic curve, where a straight line can be drawn and the fractal dimension [FD] calculated from its slope (Fig. 3). Losa and Nonnenmacher [6]. Defining a "scaling range" appears an inescapable requisite for assessing the fractality of every biological element. While the practical evaluation of the fractal dimension could be obtained by various quantitative approaches, the most reliable method is by far the box counting easily based on counting of the nonempty boxes N at a variable grid length (ε). Döllinger et al. [7]. It is obvious that the fractal theory is in opposition to the ancient, conventional vision based on Euclidean geometry and to its widely adopted concepts, such as homeostasis, linearity, smoothness, and thermodynamic reversibility, which stem from a more intuitive but artificially

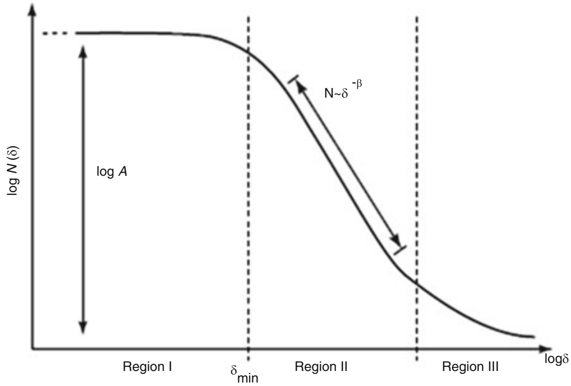

Fig. 3 The three typical regions of a biasymptotic fractal: the fractality is confined to the region II (From Döllinger et al 1998)

ideal view of reality. In the chapter of his work entitled < Epilog: The Path to Fractals >, Benoit Mandelbrot wrote "The reader knows well that the probability distribution of fractals is hyperbolic, and that the study of fractals is rife with other power law relationships." Although Mandelbrot's famous seminal paper on statistical self-similarity and fractal dimension dates back to [8] and the first coherent essay on fractal geometry was published earlier, Mandelbrot [9], it is worth here recalling that the "heuristic introduction" of this innovative discipline or, more vividly expressed "the irruption of fractal geometry" into the life sciences such as biology and medicine, Belaubre [10], actually took place in the early 80 years of the last century. Paumgartner et al. [11]. A critical review of fractal concepts was recently addressed encompassing the definition of dimensional imbalance, the modified capacity dimension and the analytical calculation of its value, the relationship with the scaling exponent, and showing that such a definition satisfies basic demands of physics, before all the dimensional balance in mathematical equations used in applied sciences. Since some concepts in fractal geometry are determined descriptively and/or qualitatively, the paper provides their exact mathematical definitions and explanations, including the Richardson's coastline method Ristanovic and Losa [12].

The Complexity of human Brain

The evolutionary concourse of two major events, "the tremendous expansion and the differentiation of the neocortex," as reported by De Felipe [13] has contributed to the development of the human brain. Today, modern neurosciences recognize the presence of fractal properties in brain at various levels, i.e., anatomical, functional,

pathological, molecular, and epigenetic, but not so long ago there was no analytical method able to objectively describe the complexity of biological systems such as the brain. The intricacy of mammalian brain folds led Mandelbrot to argue that “A quantitative study of such folding is beyond standard geometry, but fits beautifully in fractal geometry.” At that time however, there was no certainty about the brain’s geometry or about neuron branching. Anatomical-histological evidence that the complexity of the plane-filling maze formed from dendrites of neural Purkinje cells of cerebellum was more reduced in non-mammalian species than in mammals led Mandelbrot to comment: “It would be very nice if this corresponded to a decrease in D (fractal dimension), but the notion that neurons are fractals remains conjectural” affirmed Mandelbrot. Since then, a wealth of investigations have documented the fractal organization of the brain and nervous tissue system, and the implication of fractals for neurosciences has been unambiguously affirmed. Among the first applications of fractal analysis to nervous and brain tissue were the pioneering studies of Smith et al. [14, 15]. These authors showed that the fractal dimension is an unbiased measure of the complexity of neuronal borders and branching pattern and of the time course of morphological development and differentiation of spinal cord neurons in culture, increasing from 1.1 for the less differentiated neuron up to 1.5 for the most differentiated cell . Moreover, power-law scaling and other manifestations of fractal and self-similar patterns in space and/or time can be identified at all levels of neural organization. Werner [16]. Further studies have confirmed that the fractal dimension correlates with the increase in morphological complexity and neuronal maturity, Bernard et al. [17]); Milosevic et al [18]. The brain consists of distinct anatomical areas formed by nervous tissue mainly composed of neurons and glial cells of distinct types. Neurons contain the axon (a long cytoplasmic process associated with the cell body, which communicates with target organs), and the dendrites (shorter cytoplasmic processes off the cell body which allow communication between neurons), while glial cells of various types have a structural physio-immunologic role as a net via their branched and unbranched protoplasmic processes. These anatomical, morphological, and physiological properties combine to create the brain’s complexity, which can only be modeled by a supercomputer, as proposed recently, Markram [19]; De Felipe [13] (Fig. 4). While three-dimensional digital reconstructions of axonal and dendritic branching are indispensable for exploring neural function, the computational approaches enabled to quantify the intricate relationship between neuronal morphology (structure) and physiology (activity) [20]. The importance of neuronal morphology has been recognized from the early days of neuroscience. Nowadays, increasing efforts are directed to elucidate the functional roles of axonal and dendritic arbors in synaptic integration, signal transmission, network connectivity, and circuit dynamics which in turn require quantitative analyses of digital three-dimensional reconstructions. Reconstructing complex neuronal branching in digital 3D format may help map brain circuitry with its billions of connections Halavi et al. [21].

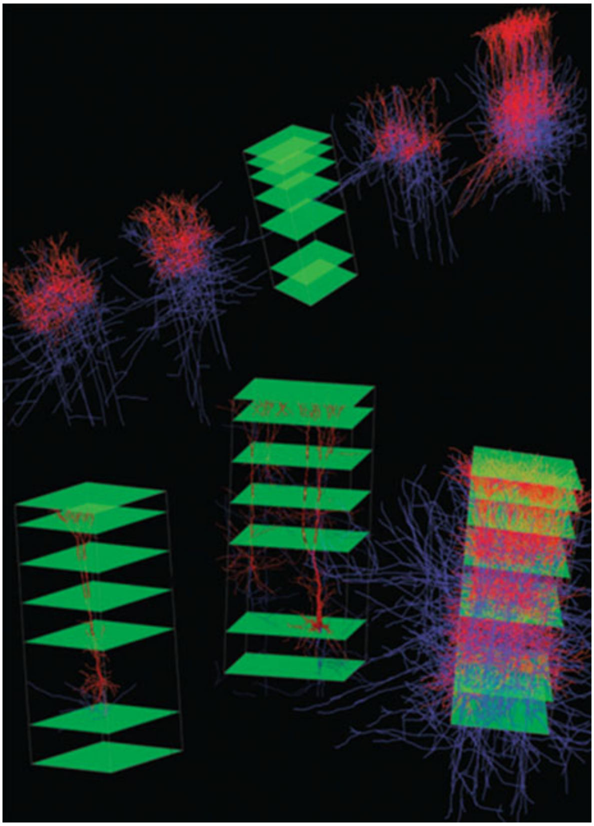

Fig. 4 Reconstructing the neocortical column. The images show the neocortical column (NCC) microcircuit in various stages of reconstruction. *Red* indicates the dendritic and *blue* the axonal arborizations. The columnar structure (*green*) illustrates the layer definition of the NCC. *Upper row*: The microcircuits (from *left* to *right*) for layers 2, 3, 4 and 5. *Lower row*, *left image*: A single thick tufted layer 5 pyramidal neuron located within the column. *Lower row*, *middle image*: One pyramidal neuron in layer 2, a small pyramidal neuron in layer 5 and the large thick tufted pyramidal neuron in layer 5. *Lower row*, *right* : an image of the NCC, with neurons located in layers 2–5. From H. Markram, 2006

Healthy and Diseased Brain Tissues

Fractal analysis was applied to anatomical/ histological images and high-resolution magnetic resonance images in order to quantify the developmental complexity of the human cerebral cortex, the alterations in diseased brain with epilepsy, schizophrenia, stroke, multiple sclerosis, cerebellar degeneration, and the morphological differentiation of the peripheral nervous system. The cortical ribbon showed a highly significant reduction of the fractal dimension in Alzheimer's Disease patients with respect to control subjects King et al. [22]. The fractal analysis has enabled to quantitatively describe the complex morphological forms in which astrocytes occur in brain of ischemic/hemorrhagic stroke and Alzheimer's disease (AD) patients Pirici et al. [23]. Fractal dimension (mean FD: 2.68) values were found higher in the Gray Matter (GM) of Multiple Sclerosis patients (MS) compared to controls (mean FD: 2.67), indicated that GM tissue in MS has higher morphological complexity, perhaps due to the presence of the inflammatory component (i.e. microglia activation) and cellular changes (synapse pruning, demyelination, brain-blood barrier changes, etc.) in the GM. Esteban [24]. In the normal human retina, blood vessels or vascular

trees exhibited an FD of 1.7, the same fractal dimension found for a diffusion-limited growth process, a finding which may have implications for the understanding of the embryological development of the retinal vascular system Masters [25]. Rat retinal ganglion cells have been classified by means of the fractal dimension Milosevic et al. [26]. Lastly, it has been shown that the quantitative evaluation of the surface fractal dimension may allow not only to measure the complex geometrical architecture [27], but also to model the development and growth of tumor neovascular systems and explore the morphological variability of vasculatures in nature, in particular the microvasculature of normal and adenomatous pituitary tissue. Di Ieva et al. [28]. The fractal analysis was recently applied on patients with cerebral arteriovenous malformations (AVM). Increased FD values related to structural vascular complexity were due to the increased number of feeding arteries in patients suffering from AVM Reishofer et al [29].

Gabriele A. Losa

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Exhibiting the Process of Science: 'The Islands of Benoît Mandelbrot: Fractals, Chaos, and the Materiality of Thinking'

Nina Samuel

I was struck . . . by the way in which many aspects of laboratory practice could be ordered by looking not at the scientist's brain (I was forbidden access!), at the cognitive structures (nothing special), nor at the paradigms (the same for thirty years), but at the transformation of rats and chemicals into paper.
(Latour [6], p. 21)

Scientific cultures are also material cultures. The world of ideas is built from stuff. With a capacity to generate and shape knowledge with their own particular logic, images are at the very core of scientific investigation.

Focusing primarily on the work of one of the most notable mathematicians of the twentieth century, the exhibition *The Islands of Benoît Mandelbrot: Fractals, Chaos, and the Materiality of Thinking* [20], explores the role of images in scientific thinking in the aftermath of a historic media shift—the new, image based society created by the digital revolution.[1] Here, the images produced by the mathematician Benoît Mandelbrot can be seen as icons of two of the most popular fields to use digital scientific imagery in the last century: chaos theory and fractal geometry. This paper presents the general idea behind the exhibition and summarizes the main arguments of its three sections *Making Visible/Sequentiality*, *Iconographies of the Invisible*, and *Drawing is Seeing*.

For thousands of years, Western thinkers assumed that the fundamental geometry of the world consisted of such regular, ideal forms as cubes, spheres, and cones. Mandelbrot, however, decided to explore the mathematics of the world not in its

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[1] The article is largely based on curatorial texts developed for the exhibition, as well as on excerpts of [20]. The exhibition was on view at the Bard Graduate Center in New York City from September 21, 2012 to January 27, 2013, and subsequently from November 16, 2013 to March 30, 2014 at Cranbrook Art Museum in Bloomfield Hills, Michigan.

Emmer M. (Ed.): Imagine Math 3. Between Culture and Mathematics
DOI 10.1007/978-3-319-01231-5_ 11,

idealized form but as it actually appears, in all its untidiness and irregularity. He studied the form of the coastlines of real islands, with all their unpredictable inlets, creeks, and furrows. Mandelbrot, in other words, looked at the world. In doing so, he flouted a prohibition in much of mathematics against the use of visual representation. To reintroduce the visual, the mathematician made extensive use of computers, transforming his discipline into an experimental science. The result was the invention of fractal geometry, a geometry of actuality rather than abstraction.

At his death in 2010, Mandelbrot left a mass of idiosyncratically organized computer printouts, films, drawings, scribble-covered manuscripts, three-dimensional objects, and photographs in his office. One year later, Mandelbrot's wife, Aliette, generously allowed access to this extraordinary trove to develop the exhibition *The Islands of Benoît Mandelbrot: Fractals, Chaos, and the Materiality of Thinking*. To explore it was like wandering through the mathematician's brain, witnessing the ephemeral traces of his very thought processes. Selections from these materials, which now are in the collection of Stanford University Libraries, form the core of the exhibition. They exemplify the practice of scientific visualization in a manner that has rarely, if ever, been exhibited before. In presenting these artifacts from an abandoned workplace the project takes seriously Bruno Latour's remark quoted above: it is impossible to look inside scientists' heads, but the material residue of their experiments, their dealings with visible things, can be studied in detail. This exhibition is based on the premise that working with material objects is an integral part of scientific thinking. If one looks at the "artifacts of science," one comes closer to obtaining a glimpse of scientific thinking than from simply reading a scientist's publications, from which the traces of thinking have generally been erased. Thinking does not take place in some ideal space devoid of objects, but through a use of materials that directly influences the shape and direction of that thinking. *The Islands of Benoît Mandelbrot* aims to explore this dependence of scientific thought on material things and images. Therefore, the exhibition does not focus on the attractive illustrations scientists produce for publication. Instead, it reveals the beauty of material thinking itself through the detritus of scientific investigation generated by experimental visual production and manipulation.

This choice is reflected even in the title, which derives from a find in Mandelbrot's office: his first fractal films (Fig. 1). They feature the jagged objects that Mandelbrot began referring to in 1975 as "fractals," shapes "loosely characterized as being violently convoluted and broken up," that cannot be understood through Euclidean geometry ([13], p. 121). The films depict such structures as animations generated by chance algorithms. One sees three-dimensional compositions—dots, lines, the relief structures produced somewhat later (his so-called "imaginary continents")—slowly rotating in front of a black background. For a few seconds, in a text that flashes between scenes, the title "The Island of Dr. Mandelbrot" suddenly appears and indicates that Mandelbrot himself associated his fractals with the 1896 fantasy novel *The Island of Dr. Moreau* [23], adding in tiny letters, "(with apologies to H. G. Wells)." With this twist on the title and the mock "apology," Mandelbrot deliberately established a connection between "science and science fiction" and, with gentle, endearing irony, related his own work

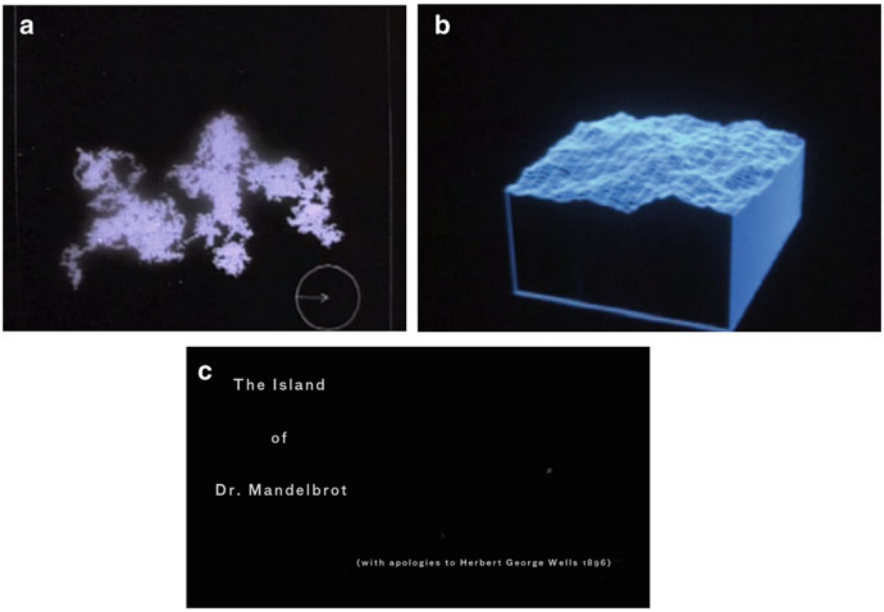

Fig. 1 Benoît Mandelbrot. Stills from computer-generated movie, ca. 1971–1975. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

to that of a long line of scientists seen as godlike, if weird, creator figures. There is a hint in these first animated simulations of a classic dream of the early pioneers of computer graphics, namely that they might, with the aid of the computer, create a new world, a second reality.

Beyond that, the exhibition concept is inspired by two different meanings of “mathematical islands” that constitute the core of Mandelbrot’s work, one figurative and one abstract, each pointing to an antipodal power of visual representation in the thinking process. The figurative meaning relates to his attempt to mathematically recreate geomorphological structures like shorelines and mountains (see section “Iconographies of the Invisible: The Imaginative Side of the Islands”); the abstract meaning refers to the story of the discovery of his Mandelbrot set and the temptation to believe the mere appearance of a picture as an unqualified truth (see sections “Making Visible/Sequentiality: The Reversal of Visibilities as Curatorial Strategy” and “Drawing Is Seeing, or: The Seductive Side of the Islands”). Pictures can release a productive, imaginative power that advances thinking, but attached to them is a seductive and potentially ungovernable energy as well, one that causes frictions between vision and thought and even, potentially, egregious errors. These two sides of pictures are inseparable and equally necessary. Accordingly, Mandelbrot’s *Islands* also symbolize the ambiguity of visual representation within the scientific thinking process.

Pictures and pictorial technologies illuminate scientific questions because they can give a new form to the invisible and pose new questions for study [1,2]. They can also have a retroactive influence on the subject as such, lending it permanent form. In this exhibition, the role of pictures in shaping the way we think about science is not only examined on the basis of Mandelbrot's working materials; it also draws on the practices of Adrien Douady, an expert in the study of dynamicals systems, as well as at the early work of Otto E. Rössler, a prominent biochemist active in the related field of "chaos theory." Beyond that, working materials from the mathematician and meteorologist Edward N. Lorenz, himself a founding father of "chaos theory," are also incorporated into the exhibition display thanks to the generous cooperation of the Library of Congress, which gave access to 16 unprocessed boxes of his notes, computer printouts, files, and photographs. One of the earliest and most important scientists to use weather simulations to study the chaotic behavior of dynamical systems, Lorenz made the study of chaos in the atmosphere a life-long occupation. Browsing through his working materials, which have been preserved the way he left them upon his death in 2008, provided a perfect complement to visits to Mandelbrot's office and to the general curatorial idea.

Making Visible/Sequentiality: The Reversal of Visibilities as Curatorial Strategy

> *When seeking new insights, I look, look, look,*
> *and play with many pictures.*
> *(One picture is never enough!)*
> (Mandelbrot [17], p. 34)

Benoît Mandelbrot and his programmers worked with sequences of pictures, where shapes appear not once, but repeatedly and with slight variations (Fig. 2). Installed as timeline according to the printed dates, we can observe how dots, patterns and shadows emerge and disappear rhythmically over time, leaving many of the pages blank—a sign of the difficulty in these early experiments of making anything visible at all. The voids in the rhythmic pattern correspond to missing pages in the sequence. Seeing patterns in these pictures helped Mandelbrot decide how to adjust the input of data. Forming an important part of Mandelbrot's thought process, these changing forms provoked new questions and led to new discoveries.

Mandelbrot's visual methodology challenged his era's accepted standards of mathematical thinking. He compared his production of many images to the work of a photographer "who takes hundreds of pictures when he or she wants to catch the right glimmer in the eye of the subject or the right light on a landscape" ([18], p. 157). We can study Mandelbrot's computer printouts as individual works, but their importance as material evidence of a thought process appears most clearly when they are placed side by side and installed as a series. Mandelbrot never published them, because it was simply not possible to make sense of them or to "clean up

Fig. 2 Benoît Mandelbrot and Mark R. Laff, programmer. “The Large White Series” (Investigations in the complex plane of iterated polynomials), 1979. IBM. Selection of computer printouts on photographic paper. Installation view from *The Islands of Benoît Mandelbrot*, New York City, 2012. Courtesy of Bard Graduate Center: Decorative Arts, Design History, Material Culture

their theory” ([17], p.157). They are intermediate steps in a materialized thought sequence. They show a new kind of imagery that challenged mathematics in the 1970s. The input of parameters and the course of the experiments were determined by perception. Suddenly, mathematical work demanded a new form of expertise for images, a pictorial connoisseurship. Observation became a new category of mathematical cognition.

The sequence of over 100 graphics from Mandelbrot’s experiments he carried out in 1979–1980 led to the discovery of the most famous symbol of chaos theory: the Mandelbrot set. Yet if these images are dominated by mere blankness, patterned by a covering of small dots, or structured by fuzzy shadows or indefinite outlines, the Mandelbrot set became generally known in a wholly different form, dazzling and brighly colored, as in the computer graphic that appeared on the cover of *Scientific American* in August 1985 (Fig. 3), which looked like a dynamic fireball racing down a black slope. Above all in the margins, the Mandelbrot set is distinguished by impressive ornamentality and self-similarity. This picture is typical of the popular composition and distribution of the set, which in the mid 1980s became one of the first scientific “icons” in the digitalization era. In the exhibition, the cover is placed behind the central installation wall and can only be seen through a round hole. The central installation wall thus embodies the conceptual nucleus of the exhibition: It showcases in an exemplary way the guiding idea of a reversal of visibilities as main curatorial strategy. What has become popular in the conscience of the general public is hidden and barely accessible; while, on the contrary, the remains of the experimental process that are normally not shown in public are put in the foreground.

Fig. 3 The popular representation of the Mandelbrot set. Scientific American 253: no. 2 (August 1985): cover. Installation view from *The Islands of Benoît Mandelbrot*, New York City, 2012. Courtesy of Bard Graduate Center: Decorative Arts, Design History, Material Culture

The installation redirects the gaze away from the final product—the cover image—and focuses on the process of discovery.

As in any experimental process, errors occurred. The exhibition also presents a collection of accidents, misprints, and undetermined images from Mandelbrot's years of research on his eponymous set that can be used to explore the questions of when and why a scientific representation may be judged to be "right" or "wrong" (Fig. 4). Some of these pictures are the result of his repeated attempts to understand the set for another class of polynomials. Mandelbrot published none of these images, yet he seemed to have valued them as they were found among his papers, and some of them are even signed.

On some printouts on the central installation wall, the later famous shape of the Mandelbrot Set can already be vaguely recognized (Fig. 5). But at this time of the experiments, Mandelbrot did not know what he was looking for, and so he remarked later: "I saw it, but I didn't know that I was seeing it" (Mandelbrot, quoted in [20], p. 34). Seeing and discovering have to be sharply distinguished as two distinct phases of sense impression: the mere availability of the new visualization tool, the computer, was not enough.

During these days of early experimentation, the computer could be called a "microscope of mathematics." The scribbled arrow on this computer-generated print (Fig. 6) point to one of the most prominent methods of investigation in this field of study: the continuous magnification of details, or "zooming," as a pictorial strategy. In the 1970s, mathematics shared image strategies and visibility challenges with the photography of microscopic biological specimens in the late-nineteenth century. In order to identify it, visualizing each specimen depended heavily on

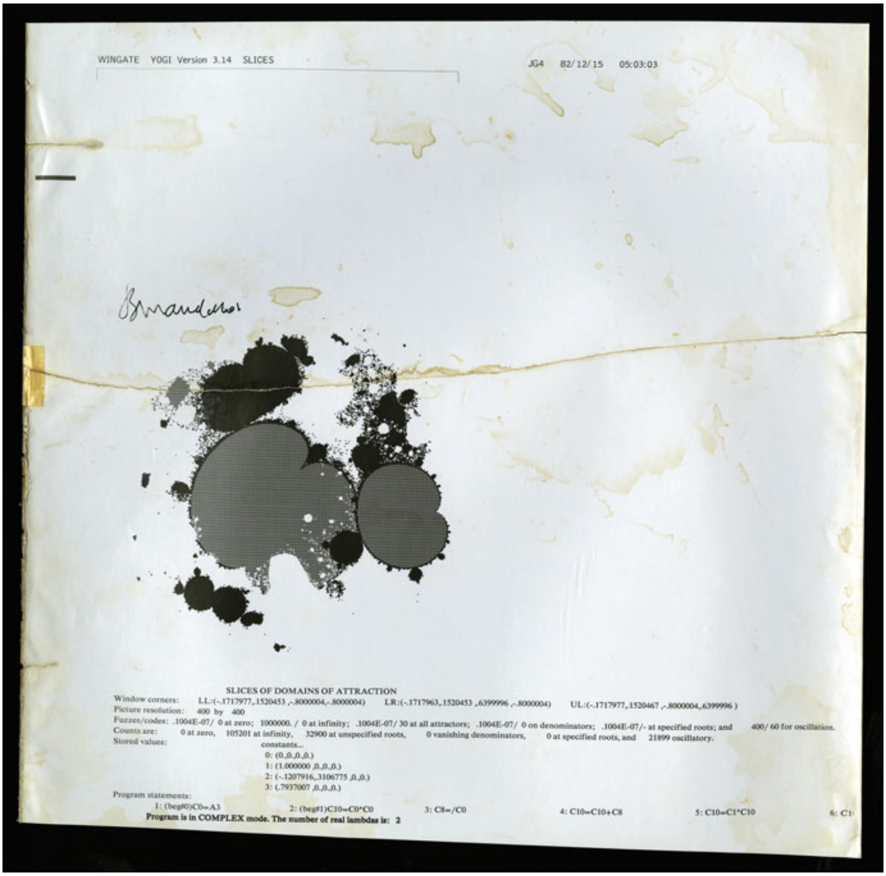

Fig. 4 Benoît Mandelbrot and Alan Norton. "Erroneous" and "destroyed" variation of the Mandelbrot set, 1982. Computer-generated print on photographic paper, torn and stained. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

deciding on a particular treatment through coloring, focusing, and creating a visual contrast between fore- and background. In a similar way, each nonlinear equation required a specific adjustment of its parameters to make its properties visible on the sheet. Differently treated, one and the same specimen or equation could yield completely different appearances. Moreover, the instrument—the optical lens or the computer—would create artifacts that were part of the picture. The condition of the image's technical production was always inseparable from the visual representation. To analyze such sensitive images, the trained eye of a specialist was needed.

This is also confirmed by the way the discovery of the Mandelbrot set took place: It were suspicious "dirt speckles"—Mandelbrot later also called them "offshore islands"—appearing on some of the pictures that seemed at first to be caused by old

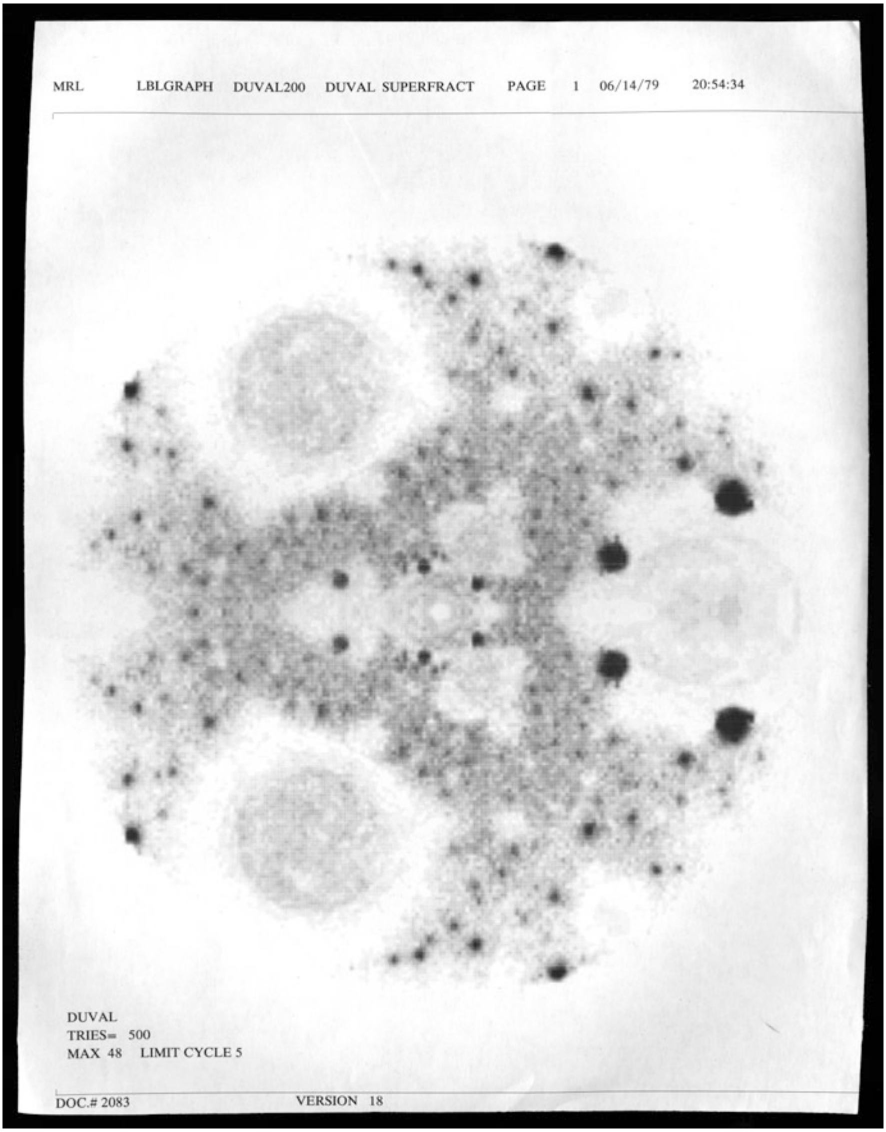

Fig. 5 Benoît Mandelbrot and Mark R. Laff, programmer. Investigations in the complex plane of iterated polynomials, 1979. Computer-generated print on photographic paper. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

printing machines ([17], p. 14). They caught Mandelbrot's attention and triggered the question of how to distinguish between picture and non-picture or phantom image: What was "dirt" and what was visualized formula? It was only through

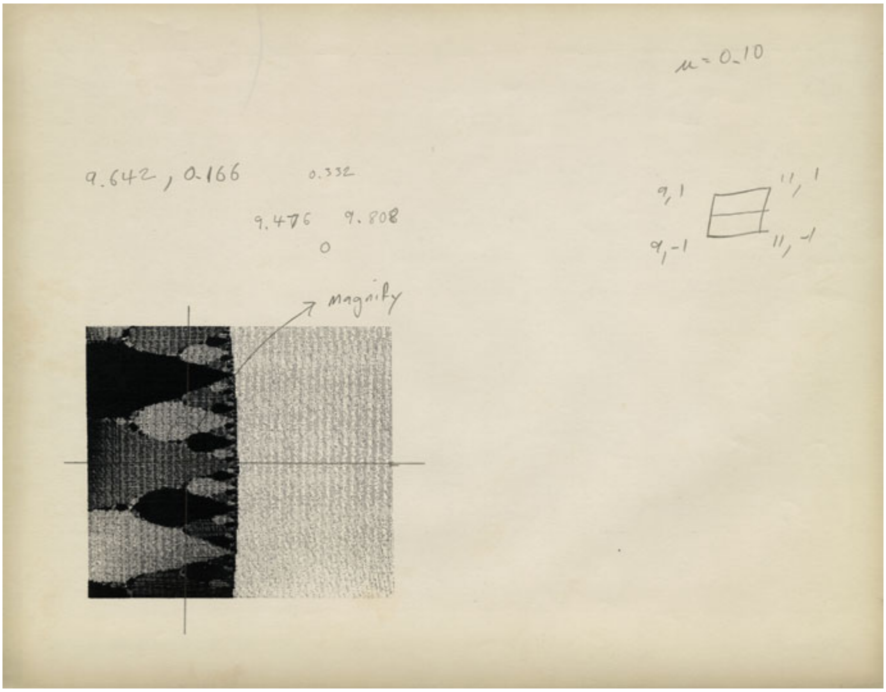

Fig. 6 Benoît Mandelbrot. Scribbled arrow (probably from a co-worker) showing the method by which details are magnified. Computer-generated print with scribbles, undated. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

zooming into these dubious picture details that the Mandelbrot set and its most characteristic feature, the self-similarity, could be discovered. This abstract meaning of mathematical "islands" is important for the whole exhibition concept and will surface again in the last exhibition chapter.

In order to write a history of chaos and fractals beyond the popular story, it is important to understand that the history of science is full of unacknowledged discoverers. John H. Hubbard might be counted among them. He was one of the first mathematicians to produce computer-graphic representations of fractals (Fig. 7), which subsequently led Mandelbrot to identify the Mandelbrot set. After meeting Mandelbrot at Princeton University in 1978, Hubbard had shown him how to produce this kind of image. Seeing Hubbard's pictures, Mandelbrot stated: "I had thought of doing these things myself, but had not mustered the courage. I was awaiting your pictures for a long, long time."[2] Instead of pixel graphics, Hubbard used ASCII (American Standard Code for Information Interchange) to visualize the equations, a hybrid of text and image derived from telegraphic codes. The

[2] Letter from Benoît Mandelbrot to John H. Hubbard, December 8, 1978.

question marks in his picture, which Hubbard referred to as "Mandelbrains," show phenomena related to the Mandelbrot set. [3] However, the technology he used did not allow him to discover the set. In the early 1980s and together with Adrien Douady, Hubbard contributed some of the most important mathematical research about the Mandelbrot set [3].

Iconographies of the Invisible: The Imaginative Side of the Islands

The eye is not specialized.

(Mandelbrot, in: Samuel [18], p. 154)

This section explores pictorial strategies and experiments that were used in various ways to relate mathematical abstractions to phenomena from the "real world" or in "nature." Mandelbrot's images not only influenced the dissemination and popularization of Mandelbrot's work, they were also part of his exploratory thinking process, helping to shape his conceptions about the mathematics expressed in the images.

Experiments with the shapes of chance—the creation of chance's cartographies—were especially meaningful for this attempt. Mandelbrot was obsessive about the pictorial possibilities of programming chance lines. In his office, this subject filled a number of boxes. Many of these images show the attempt to generate figurative cartographies of chance to stimulate the viewer's imagination. Mandelbrot gave many of these early chance pictures the name "Lévy Flight" (Fig. 8) in homage to one of his most important teachers, the French mathematician Paul Lévy, who had made important contributions to the field of probability theory in the 1940s [7]. The use of chance was one of Mandelbrot's most important strategies in creating a connection between his equations and nature.

Chance is synonymous with the unpredictably changeable, the causally incomprehensible, the incalculable, the fluctuating: Clouds are ideal symbols of a chance absence of form in constant change. These inherently un-geometrical objects became the prototype of Mandelbrot's mathematics. Etymologically, the word "fractal" can refer to a basic meteorological concept that takes into account both the rapid movements of clouds and their ragged morphology ([5], p. 231). Fully cognizant of this definition, Mandelbrot also called his geometry a science of "non-shapes"—a "morphology of the amorphous" ([15], p. 1)—and in doing so once again set it apart from Euclid's geometry.

Mandelbrot's working materials revealed that he did not only produce images, but also collected and classified images according to iconographical themes. He collected various kinds of cloud pictures in a special file (Figs. 9, 10, and 11):

[3] Interview of the author with John H. Hubbard, February 10, 2005.

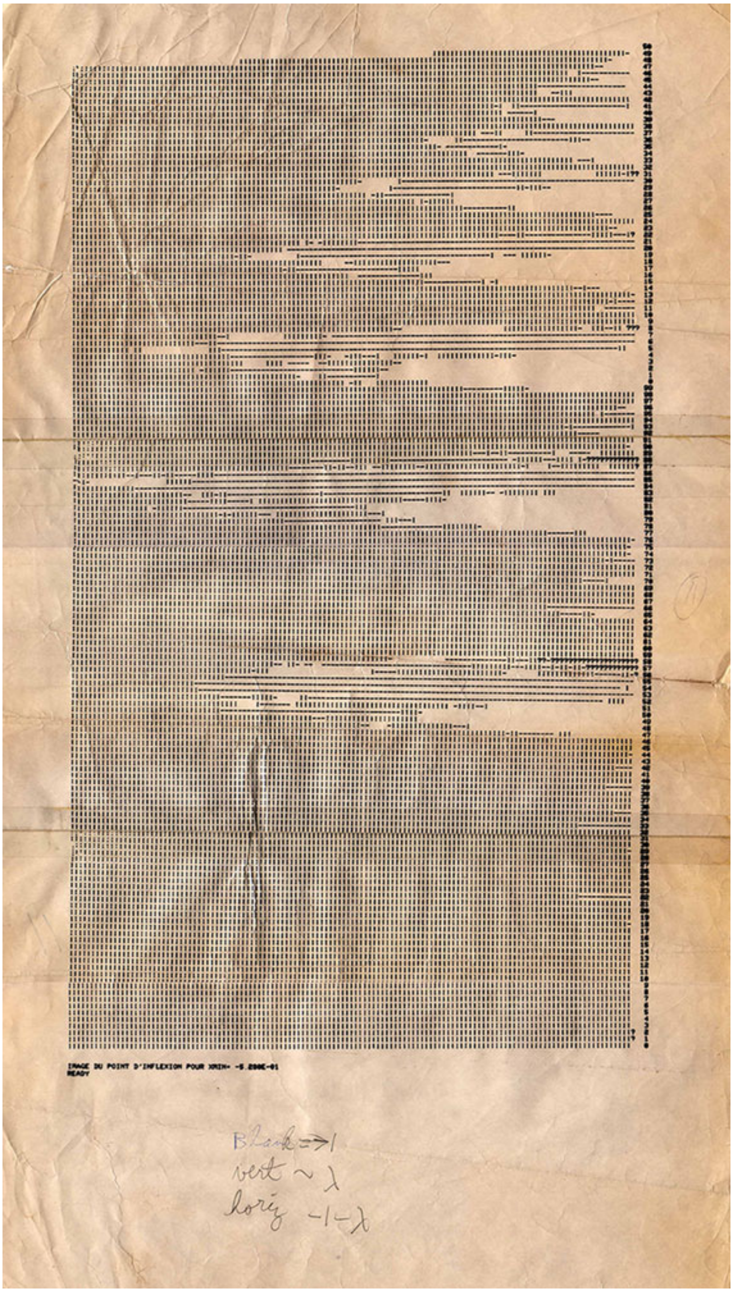

Fig. 7 John H. Hubbard. Investigations in the complex plane (Newton’s method), ca. 1977. Computer-generated print. Courtesy of John H. Hubbard

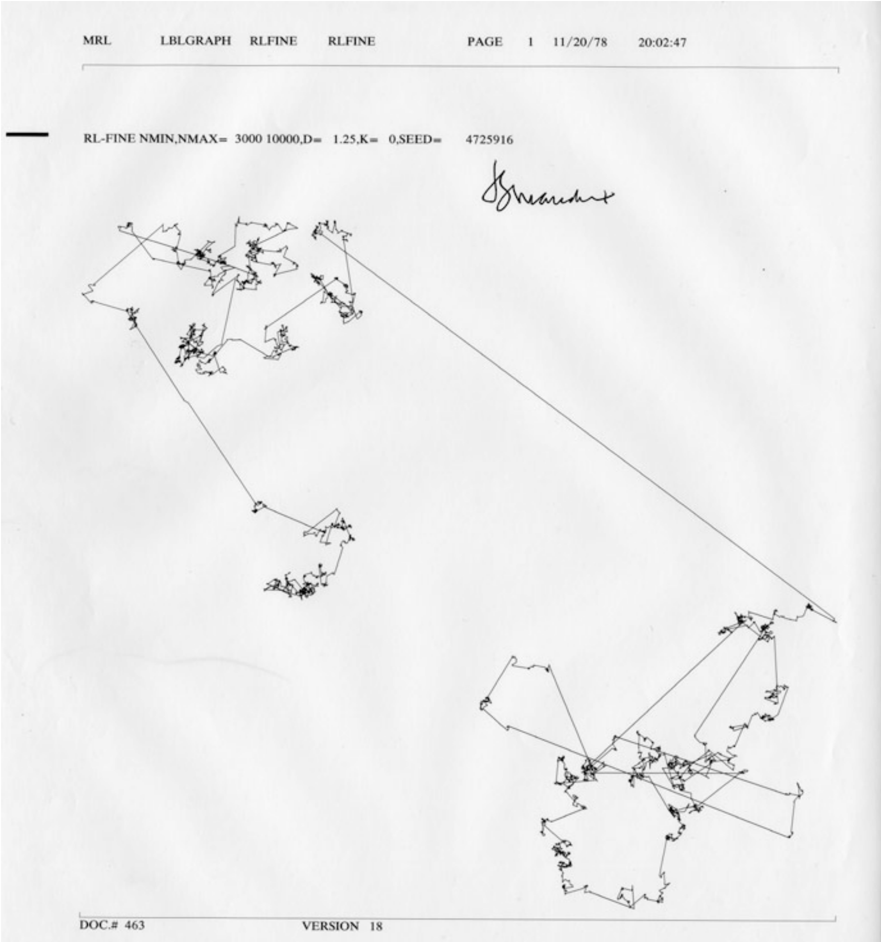

Fig. 8 Benoît Mandelbrot and Mark R. Laff, programmer. Lévy Flight, 1978. Computer-generated print on photographic paper. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

a private travel photograph of his wife's ("California 1976"), a stock-exchange advertisement with a smoking volcano cut out of a magazine, and the high-contrast, blue-and-white computer simulation by his co-worker Shaun Lovejoy [8]. Mandelbrot related these various levels of reality to each other and compared them side by side. Recalling Aby Warburg's "Mnemosyne" atlas [22], Mandelbrot's accumulations of images were based on formal analogies and combined meanings and visual traditions from different contexts in order to update the centuries-long history of the geometrization of nature. The exhibition consciously displays these assemblages in the arrangement found in the office, to preserve an order that will be lost as soon as the material will only be accessible through an archive. The logic of

Fig. 9 Aliette Mandelbrot. California summer 1976. Photograph. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

the archive is opposed to the logic of research and discovery. By creating analogies across disciplines Mandelbrot suggested both that there was a universally valid principle behind all the various pictorial phenomena and that processes could be described mathematically. The universality of the line interrupted by chance, which also characterizes the margins of clouds, stood for the universality of his fractal theory.

Mandelbrot once characterized the aim of his scientific life’s work as the restoration of a “concrete meaning” of sight in which the seeing eye is accorded the status of a scientific instrument ([17], p. 3). The reason it was such an effective instrument, he believed, was that it functioned in an interdisciplinary way. To Mandelbrot, this was owing to one of the main epistemic merits of seeing: “The problem with calculations is that they become very much specialized in the different fields. On the contrary, the eye is not specialized. It is a universal tool” (Mandelbrot, in: [18], p. 154). That each academic discipline practices its own calculation method and relies on specific knowledge acquired over centuries often renders it barely accessible even to neighboring disciplines. This could be alleviated, he insisted, through the cultivation of seeing, a theory he was fond of citing retrospectively to explain his own interdisciplinary approach and the universal validity of his fractal theory: They were, he claimed, achievements of the “unspecialized eye.” To him, formulas and pictures were inseparable: “[E]very formula evoked in my mind a completely spontaneous shape” (ibid.). But this did not imply that the eye of a “layman” was what was required; rather, he was claiming that seeing had no need of specialization to function as an instrument of understanding.

Fig. 10 "In its own quiet way." Advertisement for Philadelphia stock exchange. Unknown date. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

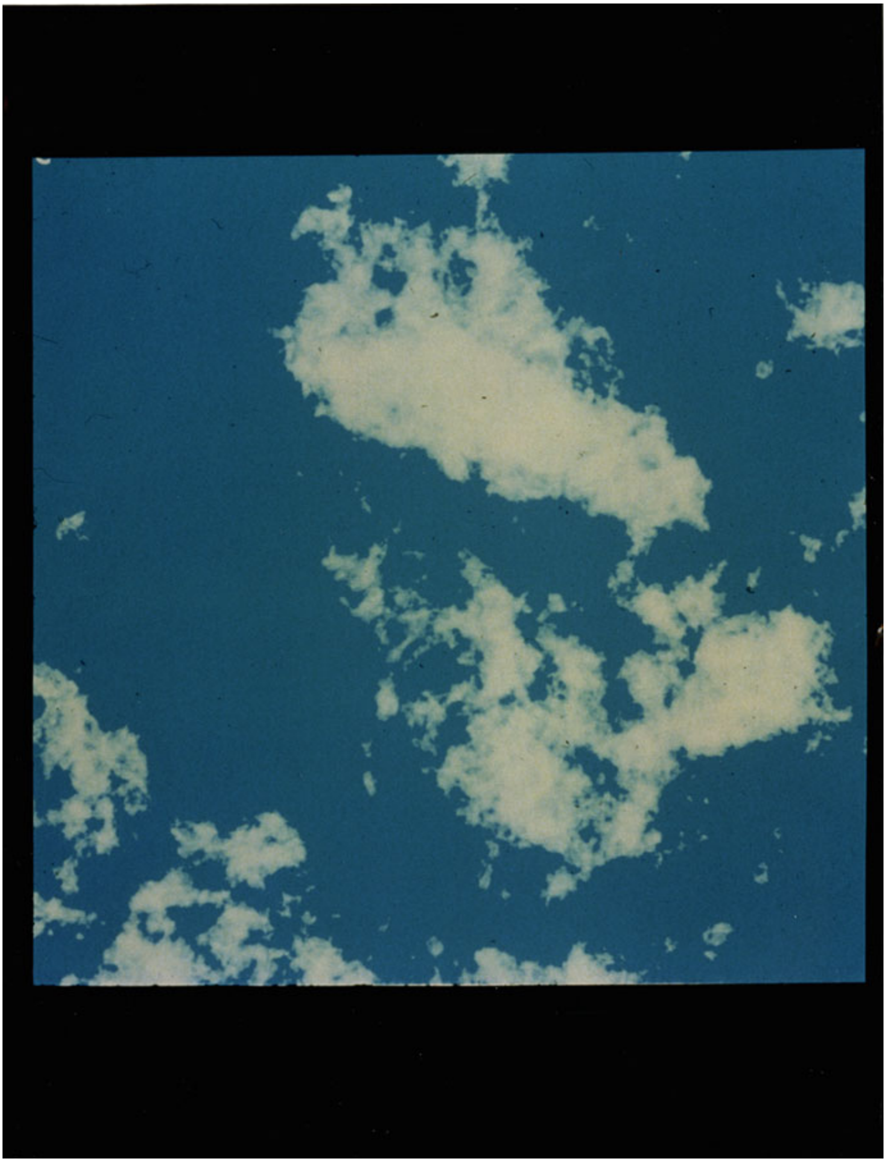

Fig. 11 Shaun Lovejoy and Benoît Mandelbrot. Image from related article [8]: "Fractal Properties of Rain and a Fractal Model," 1985. Photograph from computer-generated simulation. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

But this raises the question of how “unspecialized” an eye should be—or can be. How does his testimony relate to his use of pictures and theoretical thinking, and what clues are provided by his working materials? At this point, it is necessary to return once again to the profound clues provided by his early reference to the Island of Doctor Mandelbrot (see Fig.1), and his interest in islands more generally. Pictures of islands and coastlines were among the most numerous finds in his office in 2011. This was not a coincidence. They were crucial to his ability to bridge the gap between mathematical figures and natural forms that had been created even before it became possible to generate images with a computer. The properties of the edges of islands had already been featured in an article he published in *Science* in 1967 entitled “How Long Is the Coast of Britain?,” which is now considered the founding text in fractal geometry [10]. There, Mandelbrot argues that coastlines are self-similar and infinite in mathematical terms, constantly revealing jagged new details under enlargement. However, it would not be until the early 1970s that Mandelbrot would have access to the technical machinery he needed to generate pictures from these formulas. In the years of the first experiments with fractal coastlines, producing pictures took all night, so that Mandelbrot’s “speckled geographies” were often thrown away next morning by computer-room operators who thought them to be misprints. Moreover, for a long time, it was only possible to generate lines, the edges of which were composed of the letters O, X, M, and W; uniform surfaces were not possible. Indeed, some of the islands in the exhibition were later amplified by hand (Fig. 12). Mandelbrot did publish similar black, irregular specks in his first book, *Les Objets Fractals* [11]. He stated that their similarities to cartography were hard to overlook: One island was reminiscent of Greenland; with a quarter turn, the other one looked like Africa; and after a turn of 180°, one could see New Zealand with Bounty Island in the mass of specks ([11], p. 116). In that same year, Mandelbrot published similar chance shapes in another context, noting that he had now discovered Greece, the mirrored Sea of Okhotsk, the Gulf of Siam, and western Scotland ([12], p. 405). Mandelbrot relied on an associative interpretive logic in dealing with such ambiguity, cheerfully rotating his images in search of familiar shapes.

Mandelbrot’s interpretation of chance artifacts can be related to a pre-digital history of seeing in the realm of art. His method was preceded by a long tradition among artists of exploiting the creative and inspirational potential of chance patterns. Prominent examples of this practice can be found as early as the Renaissance: In describing the proper practice of artists, Giorgio Vasari referred to the early stage of composition when artists would make a “first drawing . . . in the form of a blotch,” which should be considered only as “a rough draft of the whole” ([21], p. 212). He advised that such blotchy sketches (Italian: *schizzi*) be “hastily thrown off” as a means of testing “the spirit of that which occurs to him” (ibid.).

As for the use of images in the construction of a fractal “geometry of nature,” Mandelbrot’s “unspecialized eye” sought to figuratively reinterpret representations of abstract knowledge. Imagination was the driving force behind this general tendency of human perception. Indeed, the way Mandelbrot saw and interpreted his images broke down traditional distinctions between the use of pictures in art

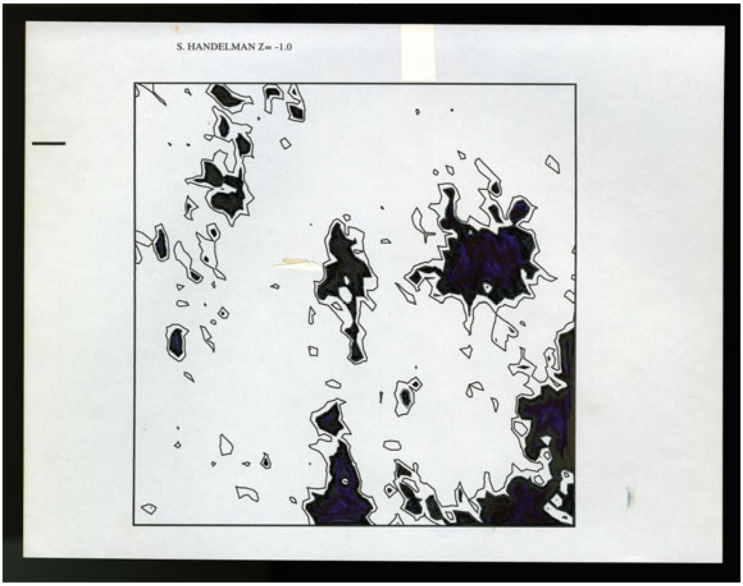

Fig. 12 Benoît Mandelbrot and Sigmund Handelman, programmer. "Islands," 1975. Hand-colored computer-generated prints. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

and science whereby images in science are supposed to be unambiguous. Instead, as Mandelbrot helped to demonstrate, pictures in the sciences have to be seen as fundamentally obstinate, intractable instruments whose vagueness can be productive in research.

This characteristic can also be observed in the series of 45 Polaroid photographs, produced in the mid-1970s by the programmer Sigmund Handelman (Figs. 13, and 14). We see different versions of a landscape, outlined in white, floating freely in black space. Although the landscape merely consists of a series of points connected by thin white lines on a black background, what we see is a three-dimensional image. When observing these images, neither light nor shadow serves as an aid to our eyes as they sense dimension and perceive space. The sequence was produced with the intention of finding the mathematical formula best suited for generating an image that we can perceive as a natural landscape. They were thus the result of aligning mathematical equations with aesthetic preferences. Varying degrees of two-dimensionality expose the experimental process. Yet it is important to realize that only some of the shapes evoke geological structures: The resemblance to nature relied on a highly selective and artificial procedure.

Fig. 13 Benoît Mandelbrot and Sigmund Handelman, programmer. Landscapes and Galaxies, series of forty-five, ca. 1974–1977. Polaroids. Installation view from *The Islands of Benoît Mandelbrot*, New York City, 2012. Courtesy of Bard Graduate Center: Decorative Arts, Design History, Material Culture

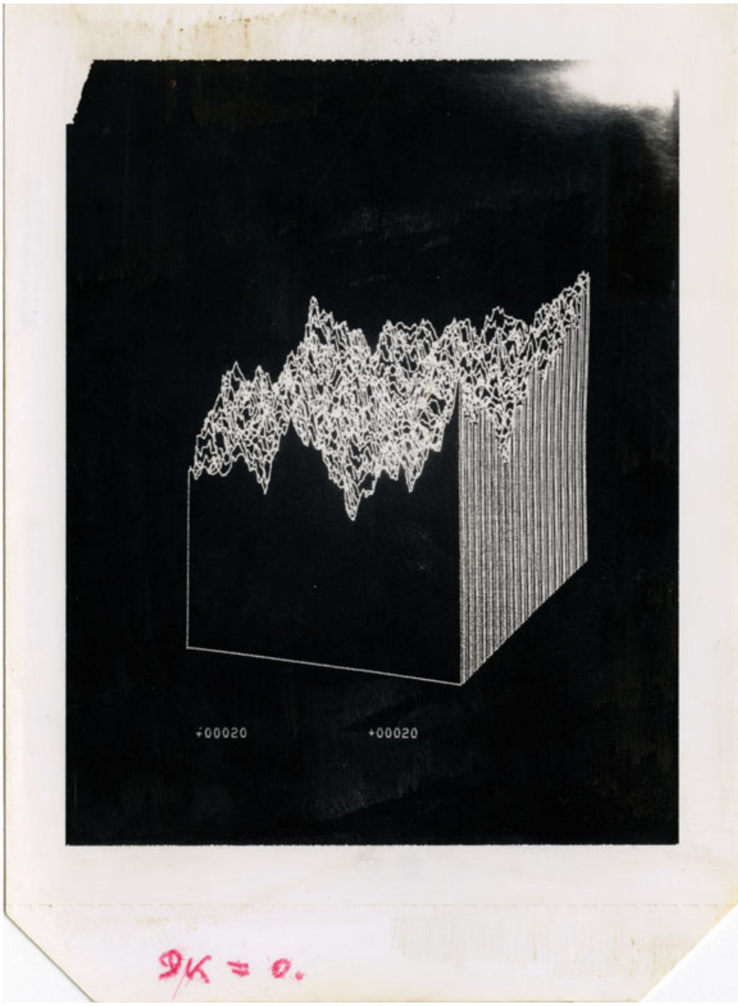

Fig. 14 Benoît Mandelbrot and Sigmund Handelman, programmer. Landscapes and Galaxies, series of forty-five, ca. 1974–1977. Polaroid. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

Drawing Is Seeing, or: The Seductive Side of the Islands

> *Drawing is a discipline of vision.*
> (Hill [4], p. 25)

Mandelbrot never tired of praising "the visual" as being crucial for thought. Unable to program a computer himself, he nevertheless came to rely so heavily on the machine as an important visual and imaginative tool that he once declared that

it had “brought back the eye into the process of thinking, research, and discovery” ([16], p. 5). This section means to challenge that claim anew: The mere availability of a new instrument alone did not achieve the “return of the eye”; the role of the hand of the scientist was also essential for the sculpting of computer data.

Sketches on paper are often the first materialized traces of an idea. Intimately linked with the thinking process, they are highly ephemeral, mostly going unseen by the public. Sometimes hastily drawn on any available sheet, sketches document the process of intellection: the sudden urge of an emerging thought taking shape; the feedback loops created between data output, paper, writing utensil, and thought; the clumsy eloquence of the material; the way reasoning needs to materialize in various ways to be effective, et cetera. Thus, while it is undeniable that the computer opened up a new realm of visibility and raised new questions, thinking often took place in a form that was very different from the glossy, digital representations for which chaos theory and fractals became known.

Examples of work by Mandelbrot’s contemporaries reveal that the production of knowledge in this field often occurs in a constant interplay between hand, instrument, and eye. A series of sketches by Edward N. Lorenz, Otto E. Rössler, and Adrien Douady provide eloquent testimony to the importance of thinking through drawing in the field of chaos theory and that this was especially valid on the threshold between analog and digital media (Figs. 15, 16, and 17).

Rössler developed his “strange attractor” in the mid-1970s via feedback-loops running between sketches and experiments with an analog computer (Fig. 15). This attractor was later called “Rössler attractor” (Rössler [19]). An attractor is a key concept in dynamics that can be described as a spatially complex geometric shape. In simple terms, attractors illustrate all the states a dynamic system can pass through over time. They are symbolic representations of long-term predictions. If they are chaotic, they are also called “strange.” Rössler used the results to produce a stereoscopic film that enabled the viewer to perceive certain shapes in three-dimensional space. One detail on these sheets is particularly astonishing. Once Rössler had become aware of the ramifications of his discovery, he marked a small spiral drawing on one of his sketches quite clearly with a green arrow—a kind of ex post facto exclamation point extolling the virtues of working out one’s thinking on paper (Fig.15).

Already a decade earlier, Lorenz had sketched his attractor in various ways before he published his conclusions in a famous paper of 1963, “Deterministic Nonperiodic Flow” [9]. This publication was a crucial inspiration for Rössler. Lorenz’ working drafts that are publicly presented in the exhibition for the first time are outstanding examples for the importance of drawing in this research field (Fig. 16). The lines drawn in different styles—bold, gently or precautiously drawn, dotted, erased—create a subtle network documenting a tentative search for a not yet clearly defined shape on the threshold between two- and three-dimensionality. They have the power to make the characteristic geometric property of chaotic attractors come

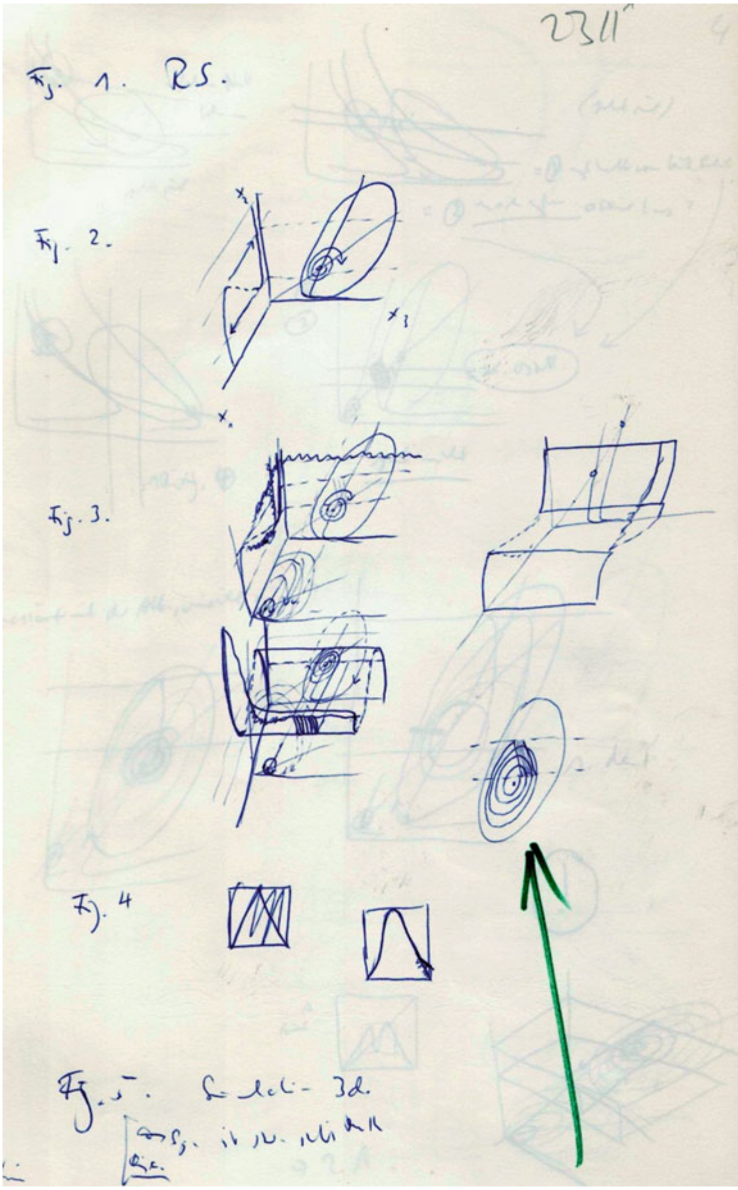

Fig. 15 Otto E. Rössler. Discovery of Rössler attractor. Sketch on paper, 1975. Courtesy Otto E. Rössler

alive and graspable: namely, to be a shape whose essential characteristic is being both surface and space at the same time—an infinitely thin, two-dimensional plane which is nevertheless spatially vaulted and infinitely folded, a shape that breaks with

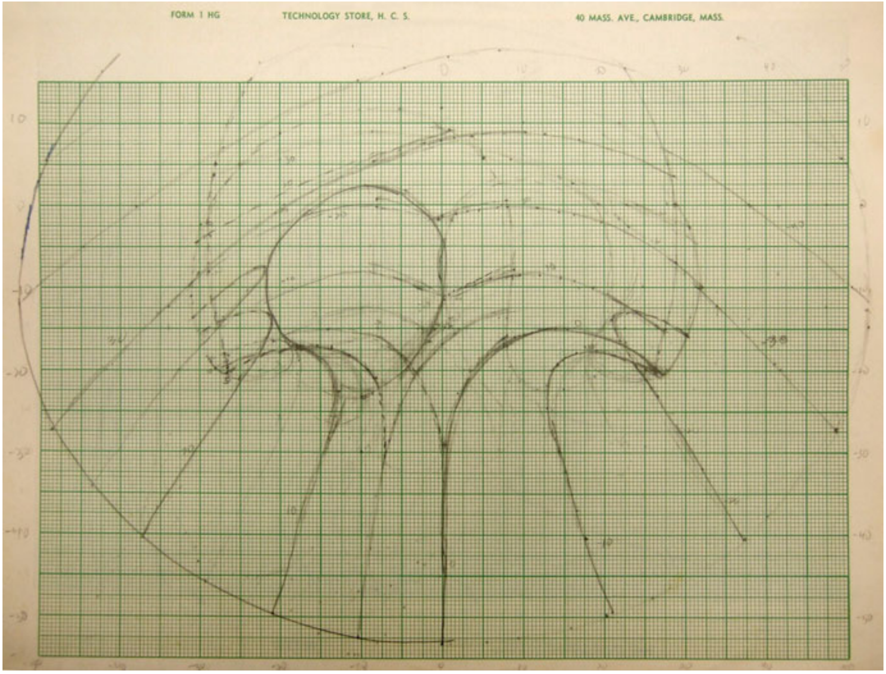

Fig. 16 Edward N. Lorenz. Sketch to explore the topography of the chaotic attractor. Poss. 1961–1962. Drawing on green scale paper. Courtesy Manuscript Division, Library of Congress, Washington, D.C

conventions of traditional Euclidean geometry and lives in-between dimensions (or, in other words, that has a “fractal dimension”). Although it is difficult to imagine these geometric properties, they are yet responsible for the practical impossibility of long-term predictability of dynamic systems, hence, for their most important feature.

Another example for the interaction between hand drawing and computer graphics can be found in the seminal work of Adrien Douady. To understand his drawings, it is important to recall the fact that computers have structurally built-in “blind spots.” What can be seen and what exists in a mathematical sense are not one and the same. In this regard, Douady made it clear that early digital pictures of fractals—the raw data—needed to go through a process of manual transformation before they could be seen with real mathematical understanding. Each of the dotted shapes on continuous paper installed vertically in the exhibition is the result of an independent numerical experiment (Fig. 17). The adjacent example on semitransparent foil with markers is one of the rare historical cases where the remodeling of this raw data through the scientist’s hand can be followed. In a letter of 2004, Douady relates the chronology of one of his discoveries, emphasizing the moment of recognition with a loopy drawing interrupting the lines of his handwriting (Fig. 17, left side). The resulting shape looks a bit like an eye gazing between the text. It can be regarded as a symbol for drawing as a process of thought in mathematics.

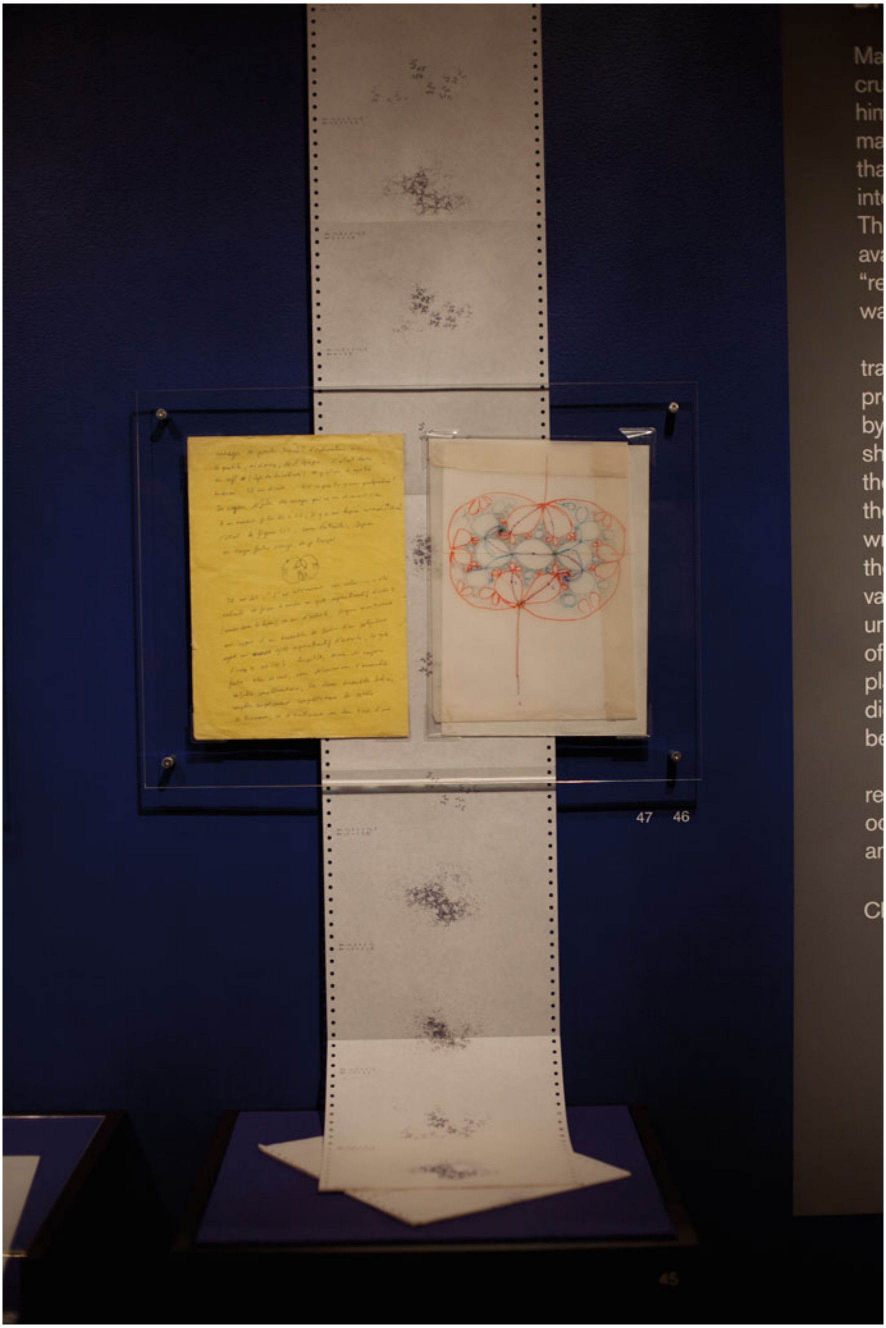

Fig. 17 Adrien Douady and John H. Hubbard. Discovery of “fractal matings” through drawing. Computer printouts on continuous paper, drawing on transparent foil and printed paper, and pen on yellow paper. Early 1980s/2004. Installation view from *The Islands of Benoît Mandelbrot*, New York City, 2012. Courtesy of Bard Graduate Center: Decorative Arts, Design History, Material Culture

Lorenz', Rössler's, and Douady's drawings have not been shown to a broader public before *The Islands of Benoît Mandelbrot*. Non-specialists often assume that ostensibly objective computer graphics best illustrate phenomena in the exact sciences. But ironically, the form of these phenomena was most likely explored by a sketching hand before they became familiar symbols of chaos.

This crucial role of drawing also leads to the aforementioned seductive potential of pictures that is in relation to the discovery of the Mandelbrot set, pointing to the second meaning of the "islands." Among the finds from Mandelbrot's office were various scribbles documenting his preoccupation with the composition of his "islands," as seen in the structure of small speckles surrounding the main bulbous shape of the Mandelbrot set seen here (Fig. 18). One of the big puzzles was whether these small details were connected with the set's central shape or not. In his first publication of 1980 in the *Annals of the New York Academy of Sciences* (Fig. 19), Mandelbrot incorrectly concluded that they were indeed "disconnected portions"

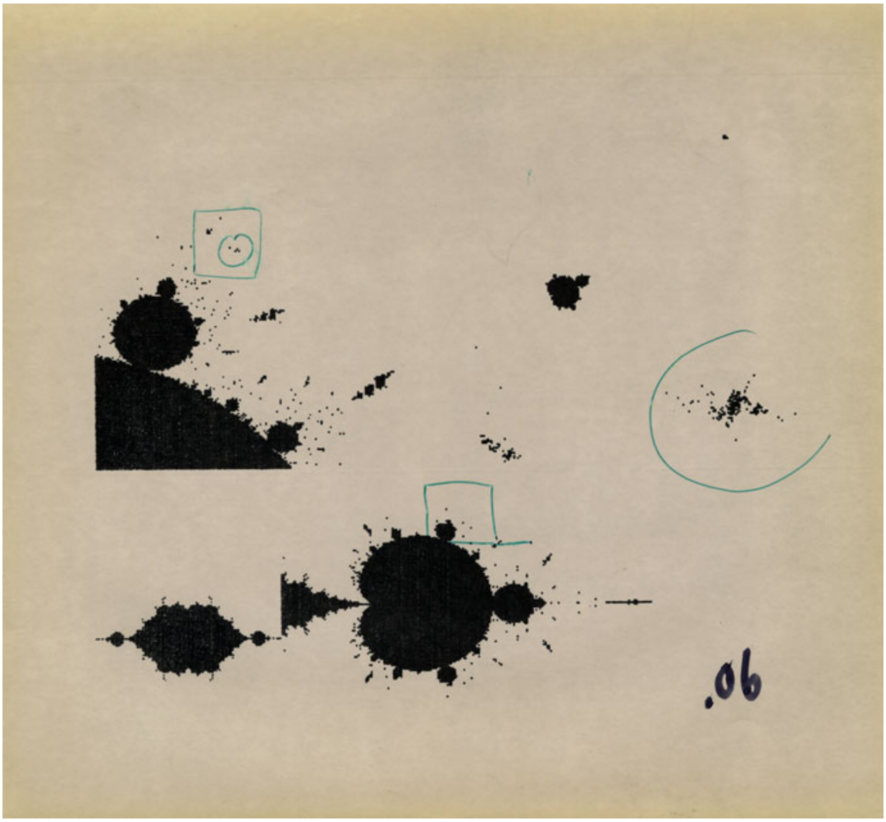

Fig. 18 Benoît Mandelbrot. Investigation of the "islands" (small speckles on the Mandelbrot set). Computer-generated print with scribbles, after 1980. Courtesy of Aliette Mandelbrot and of Department of Special Collections and University Archives, Stanford University Libraries

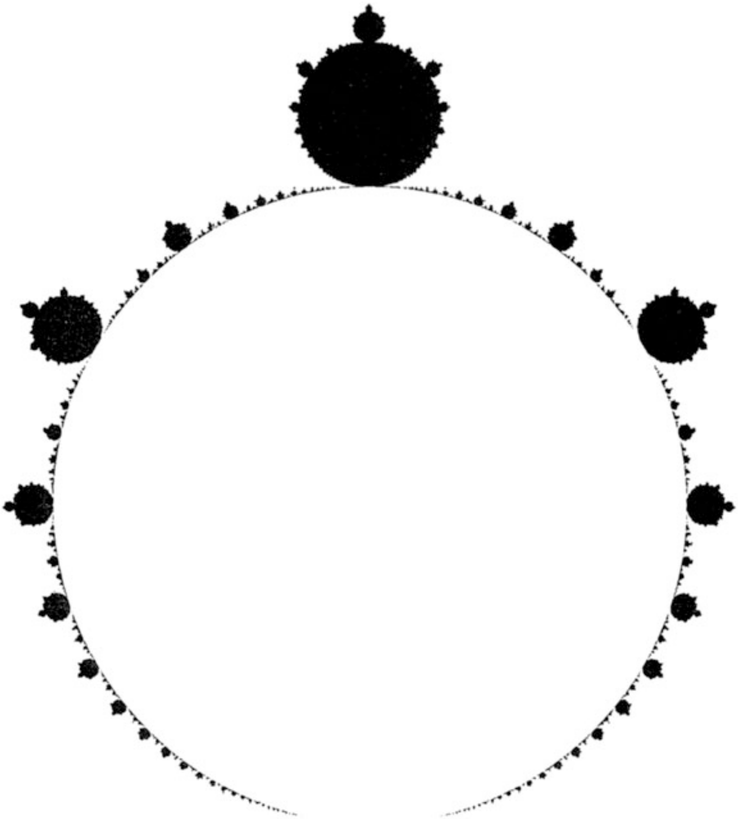

Fig. 19 Benoît Mandelbrot. Complex-plane map of the λ-domain Q. First publication of Mandelbrot Set in the *Annals of the New York Academy of Sciences*, 1980 ([14], p. 250). Surrounding specks (called "islands") had been removed mistakenly during the printing process. Courtesy of John Wiley & Sons, Inc

[14]. By confusing an apparent (but misleading) pictorial detail with mathematical evidence, Mandelbrot became a victim of his own dictum that "seeing is believing" ([15], p. 21). The imaging technology available to him limited his insight into the formula.

The displayed series of two versions of Mandelbrot's first publication together with his reproduced notes document the importance he attached to those small island speckles. Observant viewers of the picture published in the *Annals of the New York Academy of Sciences* can not even make out the specks discussed in the text (see Fig. 19), for once again they had been deleted as supposedly unwanted printing flaws by the editor: "Horrors! It is now free of specks! . . . Clearly, gremlins in the printing business had . . . repeated that evil deed" ([17], p. 22). In offprints of the article that he sent to colleagues, Mandelbrot reinserted the deleted picture details by hand

(ibid., p. 37, and see image in [20], p. 40). Since he did not see any connecting lines and had unlimited faith in his computer, he did not believe they existed, a faith he reinforced in the drawing. Suddenly, Mandelbrot's "seeing is believing" had changed to "drawing is believing": to trust in the digitally produced shape, which could be—and had to be—authenticated in drawing to exist. By means of his own successive acts of reinterpretation, the same pictorial details mutated from a hint to a proof and finally ended up as the main result: A material *trace* became mathematical *evidence*.

Nevertheless, the story of the Mandelbrot set is not one of confusing pictures with reality or of subjective naiveté. On the contrary, it raises the question of how one arrives at understanding through computer graphics in a paradigmatic way. It clearly shows how difficult it can be to maintain a distance between what the eye can see and what can be proved analytically. As the story of a struggle for agreement between observation and theory, it illustrates equally clearly how necessary it is for the eye to be schooled on pictures and how risky thinking inspired by pictures can be. In short, it says something about the fundamental ambiguity of visual perception which constitutes a main argument of the exhibition.

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Part III
Mathematics, Architecture and Design

A Topological Fluid Architecture

Michele Emmer

Venice: A Topological Metaphor

> The threads' mode of operation is simple. First, each thread is just a connector between two terms. Although more could have been found in the case of this exhibition, three were identified: Hand to head, Space to Sound, and Fountains to Neon . . . The threads are not exhaustive—rather than pretending or intending to constraint the constantly surprising open endedness of Nauman's work, they are open categories.
>
> The threads are imaginary, evolving paths among individual works, taking the viewer visually, experientially, and conceptually from one term to another, as if they were metonymically linked. More properly, the relation between the elements of one thread could be described as *topological*, in the sense that it always seems possible to imagine a passage from one to the other by stretching and twisting, shrinking and contracting, to quote words used by Nauman, in the text he wrote to accompany *Flayed Earth Flayed Self* (1973) [1].

Venice, 2009, International Art Biennal, north American artist Bruce Nauman' exhibition by the title of *Topological Gardens*: [2]

> An exhibition that is possible to imagine, is one whose structure allows an active relation with both its subject and its context. In trying to produce such an exhibition, we have used the model of *topology* to propose a specific, contextually bounded way of approaching Nauman's practice and also of interpreting the urban structure of the city in which the exhibition is set . . . *Topology* is used to establish these connections, to poke into these seemingly discrete territories so that their mutual resonance engenders a more intense relation between them and the audience.

Venice, a labirynthical, topological city undergoing daily transformation and mutations through its tidal waves, its structure continuously modified.

> As the exhibition intends to suggest that the relationships among mediums and disciplines that comprise Nauman's work are also *topological* in nature, each of the sites allows for

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> both the widest possible variety of formal and conceptual solutions to the problems posed by the different threads and the inclusions of works from the longest possible span of the artist's career. [3][1]

Erica F. Battle adds: "Repetition, permutation, redrafting, rethinking. These are keys to Nauman's practice. For this reason we position *Topological Gardens* as just one permutation of a universe that remains open to the potential for alternate axiomatic readings." [4].

Nauman had started as student at mathematics at the University of Wisconsin at Madison: [5]

> I think that I maybe knew a little bit about mathematics, I had a really good feeling about—not practical mathematics, but about understanding the structure of mathematics, a lot of interest in it, so I continued to do that. It had to do with the way is was taught, I think. I had a very small class for a couple of years and a particularly good instructor who was an algebraic topologist and presented things very well.

At the third year he abandoned his studies of mathematics to pursue art.

> Even once the academic switch was made, Nauman retained his interest in mathematics . . . Both direct mathematical concepts (such as proofs, permutations, and skew lines) and their metaphoric applications surface continuously in Nauman's artistic process. [6]

Basualdo adds that [7] "Michael Auping too cites Nauman's interest in *topology* as a math student in his essay for the catalogue of the *Raw Materials* exhibition at *Tate Modern*, London, in 2005. In Auping's account, *topology* is useful to explain the way in which Nauman navigates different mediums and disciplines in his work *twisting without tearing an idea of form*."

Transformations, metaphors, mutations, topologies: these are words that come to mind regarding back about one of the most famous contemporary artists. And Venice is the place par excellence for transformations and labyrinthine topologies, as emphasized by Basualdo "it is Venice that makes us think of Nauman's work as *topological*, but, symmetrically, it is the experience of Nauman's work that allows us to make sense of the city. They seem to correspond to each other inevitably, like a pair of hands."

These hands are a main subject of many of Nauman's works.

> Only the early plans in which the islands' contours primarily result from the demarcation of the waterways around them—a scheme more relevant the map's original users, whom it allowed an understanding of the navigational routes in and around the city—express the *topological nature* of the urban space to the same extent . . . Egle Trincarato underlines a willingness on the part of the cartographers to assign equal importance to the built city and

[1] The locations of the exhibition *Topological Gardens* were:

- U. S. Pavillon at the Giardini della Biennale, Venice.
- Università IUAV di Venezia at Tolentini, Venice
- Exhibition Space, Università di Ca' Foscari, Venice

Fig. 1 Jacopo de Barbari, *Map of Venice*, published by Antonio Kolb in 1500, engraving on six wood blocks, 132.7 × 281.1 cm)

> to the water surrounding it, as if tacitly but firmly declaring that in Venice no distinctions should be established between land and water . . . The *fluidity* of Venice is as passing as a gesture, and seen equally to imply an unfulfilled promise of meaning. [8]

Furthermore, underlining the importance of the link with topology: "*Topological* concepts and methods underlie much of modern mathematics, and the *topological* approach has clarified basic structural concepts in many of its branches . . . The urban structure of Venice might be better understood through *topology* than a map, since the representational language of a map is restricted to the abstract and simplified language of two dimensionality.

A bird's-eye view of Venice like de' Barbari's, with its stress on the morphology of the urban fabric, could be described as *topological* in relation to an orthographic projection, where the shapes of buildings and the relations between different parts of the city are sacrified to exact measurements and scale . . . Venice seems to call for an emotional *topology* instead of a map—a *topology* that could correspond to the work of an artist." [9] (Fig. 1).

On De Barbari's see the article by Guglielmo Zanelli in this volume [10] and the volume, also by Zanelli, *Navi*, *squeri e traghetti* [11], published on the occasion of the conference in Venice *Mathematics and Culture* in 2013 [12]. It was hardly a coincidence that the Biennale of Architecture in Venice in 2004 was titled *Metamorph*.

> Many of the great creative acts in art and science can be seen as fundamentally metamorphic, in the sense that they involve the conceptual re-shaping of ordering principles from one realm of human activity to another visual analogy. Seeing something as essentially similar to something else has served as a key tool in the fluid transformation of mental frameworks in every field of human endeavour. I used the expression *structural intuitions* to try to capture what I felt about the way in which such conceptual metamorphoses operate in the visual arts and the sciences. Is there anything that creators of artefacts and scientists share in their impulses, in their curiosity, in their desire to make communicative and functional images of what they see and strive to understand?

> The expression *structural intuitions* attempts to capture what I tried to say in one phrase, namely that sculptors, architects, engineers, designers and scientists often share a deep involvement with the profound sense of involvement with the beguiling structures apparent in the configurations and processes of nature—both complex and simple. I think we gain a deep satisfaction from the perception of order within apparent chaos, a satisfaction that depends on the way that our brains have evolved mechanisms for the intuitive extraction of the underlying patterns, static and dynamic.

These are the words of Martin Kemp, an art historian specialized in the relationship between art and science in the article *Intuizioni strutturali e pensiero metamorfico nell'arte, architettura e scienze*, in *Focus*, one of the volumes that make up the catalogue of the 2004 Venice International Architecture Exhibition [13].

In his article Kemp writes mainly about architecture. The image accompanying Kemp's article is a project by Frank O. Gehry [14], an architect who obviously cannot be overlooked when discussing modern architecture, continuous transformation, unfinished architecture, and infinite architecture. Kurt W. Forster, curator of the exhibit, discusses the great complexity, the enormous number of variations developed through essential technological innovations, the continuous surfaces in transformation. He cites the mathematician Ian Stewart's article entitled *Nature's numbers: discovering order and pattern in the universe* (1995). Some key words: pattern, structure, motif, order, metamorphosis, variations, transformations, mathematics [15].

Forster writes: "Recent buildings predicated upon continuous surfaces make clear that they depend in conception and realization on the use of computer technology. The single hinge upon which they turn is the computer. Any number of hybrid transformations and exchanges between traditional methods and rapidly developed software have multiplied and modified the process of elaboration and realization of projects. Hardly a method that cannot be integrated within the 'loop' of numeric calculations, but more consequential than the flexibility of elaboration and the constant back-and-forth between image and object, is the fact of architecture's migration to the realm of the virtual and simulated."

Forster continues regarding Gehry:

> What really interests Gehry is the process, in the sense of dynamic process used to achieve a structural and aesthetic result.

These words, projects, and ideas at the 2004 Exhibition were visually closely connected to the ties between mathematics, architecture, topology, and transformation. The layout of the pavilion of the Venice Exhibition was assigned to two famous

architects: Hani Rashid and Lise Anne Couture. In an article for the catalogue entitled *Asymptote, the Architecture of Metamorph*, they summarized their project as follows:

> *Asymptote*'s transformation of the *Corderie* in the *Arsenale* emerged from computer-generated morphing animation sequences derived from utilizing rules of perspective geometry with the actions and dynamics of torquing and stringing the space of the *Corderie*. The experience of *Metamorph* is spatial in that it is itself an architectural terrain of movement and flow. The exhibition architecture—from installation and exhibition design to graphic identity and catalogue design—provides for a seamless experience that fuses the Arsenale, Giardini and Venice, making explicit a contemporary reading of architecture where affinities and disparities co-mingle to produce the effects of flux and metamorphoses of form and thinking [16].

One of the studies of the layout was described quite significantly as follows: "Study of the *topological surface* that develops in the space of the Corderie and determines the movements and the curvatures used in designing levels."

Let's backtrack a bit, to the early 1990s. In 1992 the architect Eisenmann (who won the *Leone d'Oro* for his architectures at the 2004 exhibition) and his collaborators projected a skyscraper in Berlin, the *Max Reinhardt Haus*. The structure of the enormous building is based on a *topological* surface, the Moebius strip. In 1993 Ben van Berkel planned and built the *Moebius House*. So these two projects held the place of honor in the large hall of the *Corderie*, as if a reminder of an important step in contemporary architecture, in the idea of transformation, of metamorphosis. An explicit reference to topology.

Also interesting is what Hana Rashid writes in the catalog of the Biennale: [17]

> With the help of computers in all its forms developments of a new architecture, an architecture influenced and modulated by the infinite and provocative possibilities offered by these technological tools, beyond the simple promise of greater efficiency and production capacity, are emerging. These new processes and methodologies associated with history, theory, conceptual thinking, experimentation and production are radically changing not only the way we see and think about space, but also the means by which we can occupy and inhabit the territory. In one form or another, it is now within the reach of artists and architects to discover and evoke digitally induced *spatial deliria* in which the merging simulation and effect with physical reality creates the possibility of a *sublime digital metamorphosis* from thought to its realization.

Until a few decades ago these were utopian projects, and many still are, as architects also have fun designing projects that will not be realized.

The Fascination of Topology

Jules Henri Poincaré held that

> the geometrical axioms are neither synthetic a priori intuitions nor experimental facts. They are conventions. Our choice among all possible conventions is guided by experimental facts; but it remains free, and is only limited by the necessity of avoiding every contradiction, and

> thus it is that postulates may remain rigorously true even when the experimental laws which have determined their adoption are only approximate.
>
> In other words the axioms of geometry are only definitions in disguise. What then are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true and if the old weights and measures are false; if Cartesian coordinates are true and polar coordinates are false. One geometry cannot be more true than another; it can only be more convenient. Euclidean geometry is and will remain the most convenient.

Poincaré, in *Analysis Situs* [18] (Latin translation of the Greek τόπος, e λόγος), published in 1895, is responsible for the official birth of the sector of mathematics which today is called *Topology*:

> As far as I am concerned, all of the various research that I have performed has brought me to *Analysis Situs*.

Poincaré defined topology as the science that introduces us to the qualitative properties of geometric figures not only in ordinary space but also in more than 3-D space. Adding the geometry of complex systems, fractal geometry, chaos theory, and all of the *mathematical* images discovered (or invented) by mathematicians in the last 30 years using computer graphics, it is easy to see how mathematics has contributed to changing our concept of space—the space in which we live and the idea of space itself [19, 20]. Because mathematics is not merely a means of measurement in recipes but has contributed, if not determined, the way in which we understand space on earth and in the universe, specifically in regard to topology, the science of transformations, and the science of invariants. There is certainly a remarkable cultural leap: construction using techniques and materials that allow for the realization of an almost continuous transformation, a sort of contradiction between the finished product and its distortion.

It is interesting to note that the study of contemporary architecture begins with the instruments that mathematics and science make possible, more than technical instruments, cultural instruments. It is important to mention that the discovery (or invention) of non-Euclidean geometry, the higher dimensions (from the fourth on) and topology, the new idea of space to summarize, is one of the most interesting examples of the profound repercussions that mathematical ideas will have on humanistic culture, art and architecture [21–24].

Courant and Robbins write in the famous book *What is Mathematics?* (1940): [25]

> The new subject, called analysis situs or topology, has as its object the study of the properties of geometrical figures that persist even when the figures are subjected to deformations so drastic that all their metric and projective properties are lost . . . At first, the novelty of the methods in the new field left mathematicians no time to present their results in the traditional postulational form of elementary geometry. Instead, the pioneers, such as Poincaré, were forced to rely largely upon geometrical intuition. Even today a student of topology will find that by too much insistence on a rigorous form of presentation he may easily lose sight of the essential geometrical content in a mass of formal detail.

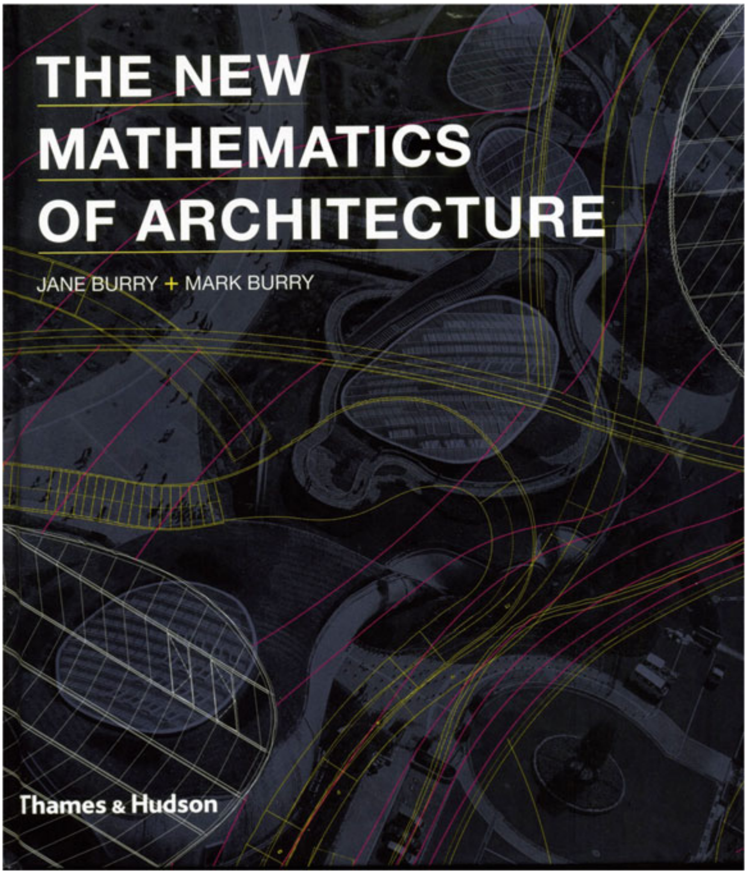

Fig. 2 Cover of the book by Mark Burry *The New Mathematics in Architecture*

The key word is geometrical intuition. Obviously over the years mathematicians have tried to bring topology into the realm of more rigorous mathematics, but there is still a strong sense of intuition involved. These two aspects, the distortions which maintain some of the geometrical properties of the figure, and intuition play an important role in the idea of space and shape from the nineteenth century to today. Some of the topological ideas were *sensed* by artists and architects in the past decades, first by artists, then much later by architects [26, 27] (Fig. 2).

Mark Burry, who is in charge to complete the Sagrada Familia by Gaudì in Barcelona dedicated a chapter on topology in his recent book [28] *The New Mathematics in Architecture*. He wrote: [29]

> The freedom that topology affords in architecture as a more generalized framework that geometry has received greater appreciation in the post-digital age . . . The essence of architectural and urban planning is also captive in such non-geometrical diagrams, as are the relationships between component spaces or activities of building. This is regardless

> of how building itself may solidify through the process of design and construction into a static, unchanging form that is also subject to detailed geometrical description. It is possible that the organization of the early development world of our childhood is a similar network of connections between significant places and things, and it is only later and gradually that the absolute reference of metrical Cartesian space is superimposed on our established perception of proximities and relationships . . .

> What is it about topology and its freedom of description that has seized modern architectural production, long after the underlying ideas were in common domain? One possible answer is the confluence of unimaged new levels of computer graphical representations with the transition of non-rational basis splines, or NURBS, from the automotive industry into other computer-aided design software . . . The dynamism of systems could not only be represented in truly dynamic models, but their manifestations could now be understood visually. Truly visual feedback changed everything. It became possible to model surfaces that could change, stretch, adopt free from curvature, or conform to a geometrical rationale without losing their integrity—wonderful surfaces that, plastically and geometrically at least, exceeded the behaviour of any known material and could be given visual material qualities at a whim . . . Topological description is being adopted as the means of mapping architectural intention, and with it arrives the progressive discovery of how to map this onto the frozen Euclidean moment in the physical world.

It is worth noting that examples of topological surfaces, in particular of the Möbius strip, have been found, in places such as the harness for the horses of the Guard of the Tsar of Russia (seventeenth century), objects that are on display at the museum of the Kremlin, in some pre-Columbian civilizations of southern Colombia, in particular the *Calima,*[2]) made for religious reasons probably. This suggests that this surface is a kind of archetypal form that is rediscovered over the centuries. It has been one of the reasons why the psychoanalyst Jacques Lacan took the Möbius strip as a symbol of his magazine *Scilicet* in the 1960s (Fig. 3).

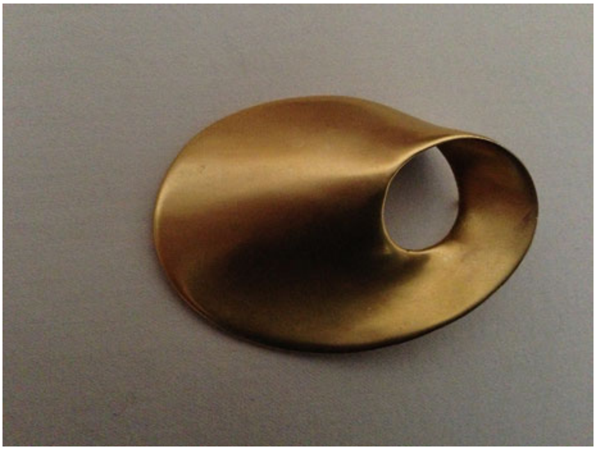

Fig. 3 Calima Culture, 200 B.C.–400 A.D. Photo by the author

[2] http://www.todacolombia.com/culturas/calima.html

It must be said that since the Biennale of 1978 topology was an important topic (and it is only one of many possible examples):

The general theme of the Biennale that year was *Dalla natura all'arte, dall'arte alla natura* (From Nature to Art, from Art to Nature) [30]. The Committee Members for the Italian Pavilion were Luigi Carluccio, Enrico Crispolti and Lara Vinca Masini. Masini defined the theme that she was concerned with as *Topology and Morphogenesis*.

Masini favoured the theme of "topological investigation as the possible identification of a mental territory that is different and alternative; an investigation that necessarily results in an ideological framework and in a critical reconsideration of history. What ensues is the recovery of anthropological memory and the practice of *citation*".

In particular, in the catalogue of the exhibition *Topologia e morfogenesi*, which was part of the exhibition *Utopia e crisi dell'antinatura* (Utopia and the Crisis of the Anti-Nature), also at the 1978 Biennale, Masini explains:

> Topology and morphogenesis are further re-examined as a taking of position in the face of a theme that is too generic and all-embracing, as well as being outdated in its literal statement, as a search for different angles from which to analyse the *territory* of art in its continual aggressiveness with respect to an actual social-political, economic and in any case existential context, albeit one that is increasingly less commensurate and conform with the life of mankind [31].

In particular, Masini's interpretation of the term *Topology* includes "in intuitive terms, that position of the impossibility of identifying an alternative topos in reality, of ascertaining the conditions of *anti-nature* in which man's panorama takes shape, today, both in the private realm as well as in his condition as a user of the city and the territory, in the increasingly evident state of alienation" [31].

Topological and Fluid Surfaces

What is more interesting is the discovery of the strip by one of the greatest artists and architects of the twentieth century, Max Bill, who told in detail on what occasion he discovered the surfaces of Mobius [32] (Fig. 4).

In 2008, a major exhibition dedicated to the works of Max Bill was held at the Royal Palace of Milan. One of the rooms was entitled *Topology*, there were a number of sculptures, but not the series *Endless Ribbon*. In the catalog, just for the section of the topological sculptures, Karl Gerstner [33] noted, after pointing out that the new spark "began 150 years ago by those who call in question a lot of things as far away as geometry" (and explicitly mention Bolyai, Lobachevskij, Riemann):

> Whoever keeps the whole artistic production of Max Bill, not only sculpture, will find concrete models, sensitive equivalent of thought patterns of modern abstract science, but in order not to create a misunderstanding: his works are not models for physics or any other teaching. They are autonomous works of art, but as with all great art, they are also

Fig. 4 M. Bill, *Endless Ribbon* 1936, from the film *The Moebius Band* by M. Emmer, 1984

a reflection of what their time embodies at the core. The spiritual underpinning of these works is in essence art and science.

In the chapter *Topological Surfaces* Alicia Imperiale writes: [34]

The architects Ben van Berkel and Caroline Bos of UN Studio discuss the impact of new scientific discoveries on architecture. The scientific discoveries have radically changed the definition of the word *Space*, attributing a topological shape to it. Rather than a static model of constitutive elements, space is perceived as something malleable, mutating, and its organization, its division, its appropriation become elastic.

And the role of topology, from the architect's perspective:

Topology is the study of the behaviour of a structure of surfaces which undergo deformations. The surface registers the changes in the differential space time leaps in a continuous deformation. This entails further potential for architectural deformation. Continuous deformation of a surface can lead to the intersection of external and internal planes in a continuous morphological mutation, exactly like in the Moebius Strip. Architects use this topological form to design houses, inserting differential fields of space and time into an otherwise static structure.

Naturally some words and ideas are changed in switching from a strictly scientific field to an artistic and architectonic one. But this is not a problem, nor a criticism. Ideas move freely and each person has the right to interpret and attempt, as with topology, to capture the essence. The role of computer graphics in all of this is essential, it allows the insertion of that deformation-time variable that would otherwise be unthinkable, not to mention unattainable.

Fig. 5 Ben Van Berkel, *Moebius House*, 1993. Photo by Christian Richters, Courtesy of UN Studio, © UN Studio

Imperiale continues regarding the Moebius Strip (Fig. 5):

> Van Berkel's house, inspired by the Moebius Strip (Moebius House), was designed as a programmatically continuous structure, that combines the continuous mutation of the dialectic sliding couples that flow into each other, from the interior to the exterior, from the activity of work to that of free time, from the fundamental to the non-fundamental structure.

Another building, another architect:

> The building, composed of arches, made up of intersecting and overlapping forms, presents a unified structure that separates, that compresses, transforms and finally comes back together on the horizontal plane at the height of the attic. The origin of the form is represented in the Moebius Strip, a three-dimensional geometric form characterized by a unique, unending surface that undergoes three iterative operations. In the first, the planes are generated from the extension of the vectors and triangulations of the surfaces . . . The second iteration overturns the strip, causing an operation similar to that in the first phase, and then appends these surfaces on top of the original form, thus creating a ghost form. The third phase applies an element of Berlinese history to the form itself, wrapping up vast public spaces between the grilled and the base of the ground floor of an already folded surface.
>
> Just as the Moebius Strip folds two sides into one surface by folding on itself, the Max Reinhardt Haus (of Peter Eisenman) denies the dialectic tradition between internal and external and confuses the distinction between public and private.

Van Berkel writes that another famous topological object, the Klein bottle, can be translated into a canal system that incorporates all of the elements that it encounters and causes them to precipitate into a new type of internally connected integral organization. Note that the words integral and internally connected have precise meanings in mathematics.

Fig. 6 Ben van Berke, UN Studio, *The Fitting Room*, 2008. © Courtesy of the ASAC, La Biennale, Venezia

But this is not a problem because "the diagrams of these topological surfaces are not used in architecture in a rigorously mathematically way, but constitute abstract diagrams, three-dimensional models that consent the architects to incorporate differential ideas of space and time into architecture."

As I mentioned before, architects became aware (albeit rather late) of the new scientific discoveries in the field of topology. And not only did they begin to design and build but also to reflect.

In the section on *Installations* of the Biennale of Architecture in Venice in 2008 UN Studio, Van Berkel, presented a large room, *The Fitting Room*, which clearly resumes once more clearly the shape of a Möbius strip. A very large, occupying one of the halls of the Arsenale entirely [35] (Fig. 6).

> The central question at the basis of the installation is: for the architectural environment what are the implications of contemporary configurations, otherwise unstable and unpredictable of life and work? . . . These layers overlap and flow into one another in a curved path made complex by projections and different lighting effects . . . The Venice Biennale is one of the last places where architecture returns to its identity as a form of art space, in which the same architecture can feel at home in the modern world.

Also at the 2008 Biennial was presented a Zaha Hadid and Patrick Schumacher's project, *Lotus* [36], it was presented in a hall of the Arsenale and by the name of *Aura*, at the *Villa Malcontenta,* one of the most famous buildings by Palladio on the Brenta River, far away from Venice (Fig. 7).

The *Aura* installation for the 2008 Venice Biennale represents a dialogue between the fluid contemporary language of the Zaha Hadid studio and the mathematical principles of harmonious architectural composition of Andrea Palladio, on the 500th anniversary of his birth. The work focuses on the piano nobile of Palladio's Villa Foscari *La Malcontenta*, which encapsulates his theory of perfect form. Accordingly, the proportions of the sequence of spaces provided the starting point for Zaha Hadid and Patrik Schumacher's study.

Fig. 7 Zaha Hadid & Patrick Schumacher, *Lotus*, 2008. © Courtesy of the ASAC, La Biennale, Venezia

Changing, transforming the rules, "instead of representing a system already domesticated through internal rules, the *Lotus* room (author's note: that of the Arsenale) seduces through the folds of undulating rhythm, its exclusions, its reconfigurability and its ability to remain outside of categories." In November 2009, a new space for contemporary art and architecture in Rome, MAXXI was inaugurated.

This how the project is presented at the site of the study of Zara Hadid: [37]

> MAXXI supercedes the notion of museum as *object* or fixed entity, presenting instead a field of buildings accessible to all, with no firm boundary between what is within and what without. Central to this new reality—its primary force—is a confluence of lines—walls that constantly intersect and separate to create indoor and outdoor spaces (Fig. 8).
>
> MAXXI integrates itself with its surrounds, re-interpretation urban grids to generate its own geometric complexity. Through the flow of its walls it defines major streams—the galleries—and minor streams—interconnections and bridges, delighting in a peculiar L-shape footprint which in this context becomes 'liberation'—a freedom to bundle, twist and turn through existing buildings. In this very meandering MAXXI both draws on and feeds the cultural vitality of its mother city . . . MAXXI expresses itself through glass, steel and cement—delighting in neutrality, achieving great curatorial flexibility and variety. To wander through, to experience this place—these spaces—is to encounter constantly changing vistas and surprises.

At the inauguration of MAXXI Zaha Hadid said that first of all she had to decide whether or not to keep all existing buildings. Once the decision, she began to study the geometries that would replaced, them orthogonal, parallel or diagonal. "What appeared was a confluence of lines of different geometrie present on the site. This way it started and a *fluid interpretation* of the space emerged." The *fluidity* is now one of the keywords of contemporary architecture. Among other things, Zaha Hadid has a degree in mathematics.

Fig. 8 Zaha Hadid, *MAXXI Museo nazionale delle arti del XXI secolo*, Roma, 2009. Foto Francesco Bolis, © Courtesy Fondazione MAXXI

The Topological Tendency in Architecture

In her 1999 doctoral thesis *Architettura e Topologia*: *per una teoria spaziale della architettura*, Giuseppa Di Cristina writes: [38]

> Architecture's final conquest is space: this is generated through a sort of positional logic of the elements, that is through the arrangement that spatial relationships generate; the formal value is thus substituted by the spatial value of the configuration: the external aspect of the form is not as important as the spatial quality. And thus topological geometry, without "measure" and characteristic of non-rigid figures, is not something purely abstract that comes before architecture, but a trace left by that modality of action in the spatial concretization of architecture.

In 2001 Di Cristina edited a book on *Architecture and Science* [39]. In her introduction *The Topological Tendency in Architecture* Di Cristina clarifies that

> The articles that are included here bear witness to the interweaving of this architectural neo-avant-garde with scientific mathematical thought, in particular topological thought: although no proper theory of topological architecture has yet been formulated, one could nevertheless speak of a topological tendency in architects at both the theoretical and operative levels. In particular, developments in modern geometry or mathematics, perceptual psychology and computer graphics have an influence on the present formal renewal of architecture and on the evolution of architectural thought. What mainly interests architects theorizing the logic of curvability and pliability is the significance of the *event*, of *evolution*, of *process*, or the innate dynamism in the fluid and flexible configurations of what is now called topological architecture.
>
> Topological architecture means that dynamic variation of form, facilitated by information technology, by computer assisted design, by animation software. The topologification of architectonic forms according to dynamic and complex configurations leads architectural

> design to a new and often spectacular plasticity, in the footsteps of the Baroque or organic Expressionism.

Stephen Perrella, one of the most interesting *virtual* architects describes *Architectural Topology* as follows [40]:

> Architectural topology is the mutation of form, structure, context and programme into interwoven patterns and complex dynamics. Over the past several years, a design sensibility has unfolded whereby architectural surfaces and the topologising of form are being systematically explored and unfolded into various architectural programmes. Influenced by the inherent temporalities of animation software, augmented reality, computer-aided manufactured and informatics in general, topological "space" differs from Cartesian space in that it imbricates temporal events-within form. Space then, is no longer a vacuum within which subjects and objects are contained, space is instead transformed into an interconnected, dense web of particularities and singularities better understood as substance or filled space.
>
> This nexus also entails more specifically the pervasive deployment of teletechnology within praxis, leading to an usurping of the real (material) and an unintentional dependency on simulation.
>
> Observations in which ideas about geometry, topology, computer graphics, and space time merge. Over the years the cultural nexus has been successful: new words, new meanings, and new connections.

These are observations in which ideas on geometry, topology, computer graphics, space-time, and so on converge. The reasons why the links, the Ariadne's threads of culture over the years have worked: new words, new meanings, new links, new influences, new forms, new spaces. In a never-ending process, in which the mathematical ideas on space have played an important part.

A Last Example: Armani Fifth Avenue, NYC, 2009

This is how it is described the incredible staircase inside the building, realized by the Studio of Massimiliano and Dorina Fuksas in 2009: [41]

> Situated in the centre of New York, the showroom develops on four different levels and it is conceived as a single space, without clear distinctions, a space in harmony connected with the power generated by the vortex that is the staircase. The heart of the building is, in fact, epitomized by the staircase. Structure in rolled calendar steel and cladded in a plastic layer that highlights its exceptional sculptural presence. It is an entity that it is almost impossible to convey in terms of any normal geometric shape that originate from a vortex with great dynamism, surrounded by the different levels ... The movement of the ribbons that constitutes the staircase ... The *fluidity* of the internal space is rendered by the wall of the continuous threads. (Fig. 9).

Fig. 9 Massimiliano e Doriana Fuksas, *Armani Store NY*, 2009. Courtesy of © Archivio Fuksas

Appendix

Of course, not all architects agree with the fluidity, the topology of the new architectures. Here are some views of the Italian architect Vittorio Gregotti written about one of my articles on fluid architecture: [42] (The insistence on the word discipline is typical, of, not all, the Italian university professors).

> Of course architecture moves in an area in which different ideologies and techniques make difference more complex, and it is precisely the misunderstandings around multidisciplinary that contributes to the serious crisis that our discipline faces today, specifically as artistic practice.
>
> This fact is evident even only scrolling the color illustrations of the article, almost all of them tragic demonstrations of the disastrous situation of the architecture, as well as the mistaken transposition of very serious research in the field of geometry of the last century, which became, so often misused in my discipline, stimulating formalistic calligraphy too sensitive to the audio visibility of our time.
>
> The transpositions in architecture (of the beautiful sculptures by Max Bill) are (at least for now) a very expensive disastrous exercise where the term fluidity becomes a calligraphy more inspired on comics that on reflexions on post-euclidean geometries.
>
> The author is mistaken proposing the example of Venice as evidence of fluidity, or focusing on Bruce Naiman or Escher, interesting visual artists but far from the objectives of architecture.
>
> Overall, the volume is certainly a valuable document, which is important for my discipline especially if it will spark a discussion on the extraordinary contribution of mathematical thinking, but also on the fatal consequences of its interpretation as an invitation to liquefaction of the disciplines so damaging for architecture of our years.

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Pasta Mathematica

The Mathematics of a Best-Loved Foodstuffs

George L. Legendre

Introduction

The fun-sounding research eventually published in the form of *Pasta By Design*[1] started off as an innocuous bet (Fig. 1). Eventually it earned this writer the sort of media coverage reserved for prime-time culture: interviews with the New York Times, chats with lifestyle hosts on US primetime cable TV (CBS on Sunday); reviews in the Wall Street Journal, the Financial Times, the US National Public Radio, the *Frankfurter Allgemeine Zeitung*, the Russian edition of GQ magazine, and even Oprah.com. In 2 years since its publication, this unique book has earned generous praise and dismissive criticism in equal measure. Given that it is part mathematical paper, the reception has been both markedly mainstream and (on occasion) emotional—two reactions we would not readily associate with scientific publishing. From author to reader via publisher, distributor, and mass media, *Pasta By Design* has appealed to drastically different sensibilities.

From the author's perspective the endeavor has been, from very beginning, a Dadaist prank. The original idea was to blend the sacred with the profane—mathematics with cooking tips—to arrive at the sort of thing Salvador Dali (of the Lobster Telephone fame) could have done next. In the early stages, the working title of the budding art project ('*the Pasta Book*') had the understated irony of pop art, best conveyed in our minds at least by a blank book jacket more suitable for office memos. The arty approach was eventually dropped when a media-savvy London publisher pointed to possible negative comments by online shoppers lured into buying a book by its title—and posting a scathing Amazon review as a result (which will happen anyway). As the final title and look-and-feel of the book suggest, our marketing-minded publishers saw *Pasta By Design* first and foremost as a design title

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[1] Pasta By Design, George L. Legendre Foreword by Paola Antonelli Thames & Hudson, London 2011. *Pasta Und Design* Springer Verlag, Berlin Heidelberg 2012. German edition Translated by Anna Schleizer.

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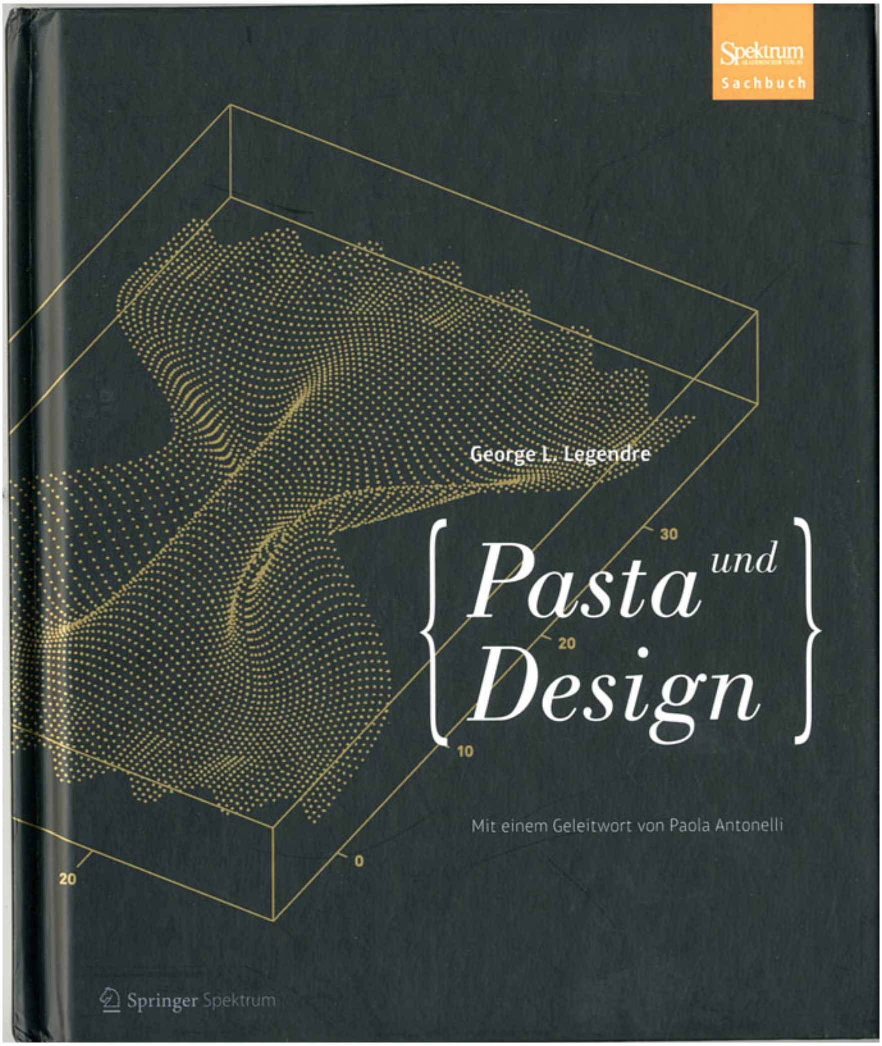

Fig. 1 *Pasta Und Design* (2013) Cover of the German edition of *Pasta by Design*. Originally published by Thames and Hudson, London 2011. Photography by Stefano Graziani. Preface by Paola Antonelli

with a black jacket, because designers wear black (unsurprisingly the publisher's vision prevailed).

Readers were divided too. Some saw *Pasta By Design* as a technical work, noting that it would either "peak the layperson's interest in mathematics", or (less charitably) provide "food porn for engineers". Both types of comments suggest a limit as to how much an art book based on mathematics can be loved, which, by and large, makes sense: our design culture has largely embraced the innovations of

computing but has a lot less time for the formulated thought lying at the very root of the breakthrough. There are many reasons behind this paradox. Mathematics is a deeply abstract discipline, and as such it is easily misunderstood. On a personal level, it is likely to summon memories of hard graft, frustration and perhaps even of failure. A book on the subject will hardly spread the love.

Pasta by Design is not about mathematics per se—it's an art project, with method to boot. Thankfully other readers saw this too and praised the work's strangeness and beauty. Soon after publication, I even received an offer by card-carrying conceptual artist Serkan Özkaya to collaborate on a 'proper' art installation of the same material, eventually commissioned by the Istanbul-based contemporary art gallery *Galerist*.[2]

Serkan Özkaya specializes in replicating original artwork in different contexts, so *One and Three Pasta* (*Bir ve Uç Makarna*) reprises the conceptualist strategy of Joseph Kosuth's iconic piece *One and Three Chairs* (1965) (Figs. 2 and 3). Kosuth's original installation, according to Wikipedia, "*consists of a chair*, *a photograph of this chair and an enlarged dictionary definition of the word* '*Chair*'. *The photograph depicts the chair as it is actually installed in the room*, *and hence the work changes every time it is installed in a new venue*".[3] Likewise *One and Three Pasta* (*Bir ve Uç Makarna*) consists of 92 shelf displays of real pasta shapes, perfect replicas in nylon, and copies of the corresponding equations as published in *Pasta by Design*.

While pondering Özkaya's reinterpretation of Kossuth, I wondered about the appropriateness of exhibiting the nylon replica next to the original equation. Since the shape had been 3D-printed directly from the formula, why bother with the output when you already have the source? Or bother with the music when you already have the score? Eventually I understood that the replica fulfilled the same role assigned by Kossuth to the photograph, i.e. to provide an iconic representation of the artifact. The equation on the other hand, fulfilled the function of the enlarged dictionary definition: a collection of symbols 'signifying' the artifact itself. In any case, Özkaya's artistic instinct was correct: the pairing of real and replica exuded an unparalleled physical aura, which no amount of print can reproduce.

Is *One and Three Pasta* (and by extension *Pasta by Design*) art? The ultimate arbiter of the question must be the art market itself, and there too reception was mixed. From anecdotal evidence, the roster of international curators who visited the gallery in 2012 seem to have either loved it or hated it, and significantly the piece has yet to sell at the time of writing. Some well-known contemporary artists praised it in full. By and large Serkan Özkaya's otherwise dedicated collectors passed on the opportunity to purchase it: to them it was either 'over the top' –or simply not to their liking. In conclusion, it is fair to say that the 'success' or 'failure' of this artistic endeavor has generally come down to people's attitude towards mathematics. When things worked out, mathematics claimed the credit; when it didn't, mathematics shouldered the blame.

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[2] The exhibition opened at *Galerist* Tepembasi, Istanbul in October 2012.

[3] http://en.wikipedia.org/wiki/One_and_Three_Chairs.

Fig. 2 *One and Three Pasta* (2013). George L. Legendre in collaboration with Serkan Özkaya. Overall view of installation, Galerist Art Gallery, Istanbul

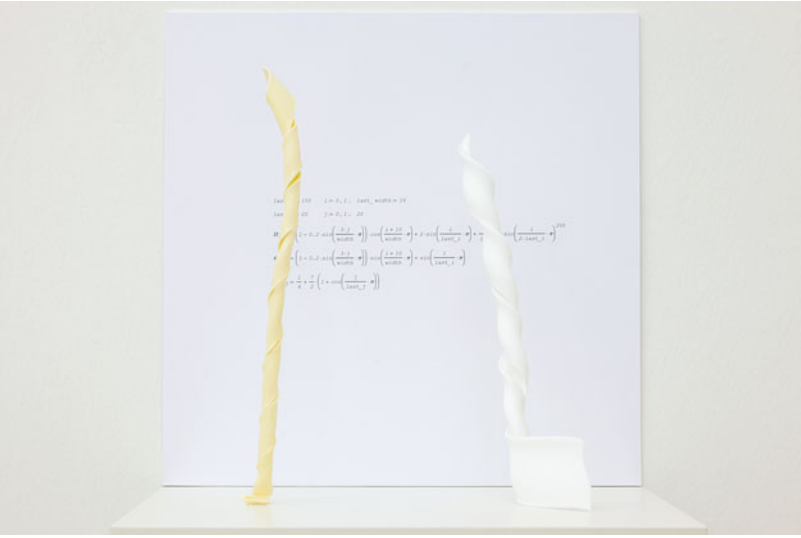

Fig. 3 *One and Three Pasta* (2013). George L. Legendre in collaboration with Serkan Özkaya. Detail of installation, Galerist Art Gallery, Istanbul

Methodology

With over 200 elaborate diagrams and photographs, as well as a few words on regional provenance and cuisine, *Pasta by Design* offers a surreal interpretation of the everyday experience of making and eating pasta. Specifically, it presents a magnified view of the *forming* stage of the pasta-making process (the mechanical extrusion of a mixture of durum wheat-flour and water into the familiar shape). Drawing on several years of practical experience—I have been using mathematics to design anything from pedestrian bridges to playground slides for a number of years—I used mathematics to model reality while making it fun.

To understand the scope of this project, it is necessary to offer a few words of explanation about our working methods. We use mathematics to *design* new things; hence our approach implies a direct recourse to generative symbols and marks. In other words, we work with symbolic equations rather than ordinary surface software commands, or symbols rather than buttons and sliders. The difference matters. Software is built by *chunking*, or consolidating lower-level steps into higher-level ones, like building a pyramid of sorts. To work directly with equations is to work, if not at the bottom, at least pretty down low, where most of the room is, but nothing is defined. We are not interested in the kind of facility you find at the top. As designers, to counter the seemingly endless multiplicity of choice, we like to stick to the low-level approach. Our workstations do not run the latest software; instead we go straight to the base materials that software is made of – the raw equations that we take for granted, which usually exist somewhere under the hood. In this sense we choose to manufacture our own constraints, forcing ourselves, metaphorically, to type wearing boxing gloves. And in the enforced lack of dexterity we can somehow circumscribe issues to be fruitfully developed. Hence, alongside our peers we are really quite 'primitive'. We do not pursue a dazzling technological agenda through the very latest facilities, but work instead at a kind of infra level—the infra-technological, the common symbolic language on which these processes are based.

Working like this comes at a price: there are design explorations we must abandon because we do not know how to write them, although, in fairness, true technical hurdles are few and far between. In the past 5 years the most complicated mathematics we wrote involved a set of calculations to conformally map a random two-dimensional curve on a random three-dimensional surface (a concept akin to using *any* pen to draw on *any* sheet of paper). So far there has been no use for this particular algorithm, which proves that harder-than-average mathematics does not make for challenging design explorations –and vice-versa.

Pasta by Design

In the twin worlds of design and engineering, mathematics is used to model tangible artifacts such as boat hulls and car bodies, or abstractions like air flows or complex financial yields (these are not always accurate). The process is usually abstruse and speaks solely to specialists. So when my Italian colleague Marco Guarnieri suggested (over a glass of chianti and a steaming pile of *spaghetti all'olio*, *aglio e peperoncino*) that we take a closer look at something familiar to everyone, we leapt on the opportunity, and the blurred outline of the 'mathematics of pasta' gradually came into focus. Our concept is simple: to figure out the mathematical formulas of one of the most popular foods on earth and to use them to produce an inventory, guide and even culinary resource on the subject.

Mass-producing pasta involves hard science. Extruding dough to produce a pasta shape, for instance, is dependent on pressure, viscosity, precise temperatures, pressure differentials and air flow: all these factors determine the material properties of the product—as well as its quality and market value. Scientists have published papers on how to synthesize the enzymes of durum wheat, the primary ingredient of pasta, and the impact of kneading and drying on the digestive tract. At this level of science, the underlying mathematics has only 'invisible' benefits: you cannot *see* the thermodynamic effects of viscosity, which are crucial to the forming of the shape, but you will probably taste them (making good pasta requires a perfect balance) (Figs. 4 and 5).

Aside from the fascinating but invisible mathematics/physics of watering, mixing and kneading flour, there is also the essential beauty of the visible mathematics of the geometry of pasta. This 'customary beauty' is critical because it allows us to appreciate each shape while rethinking the variety of pasta at large. Exploring this 'customary beauty' is the main ambition of Pasta By Design. Using a mathematical surface model originally developed for building structures, I devised a magnified view of the forming stage of the pasta-making process (the mechanical extrusion of a mix of durum wheat flour and water into the familiar edible shape), and interpreted it in the form of a conceptual 'pasta family tree'.

The concept of a 'pasta family tree' is especially important because the canon of pasta is both open and large. Many regions in Italy produce lesser-known varieties, or spawn minor variations of established ones, then give them local names. This makes classification a particularly difficult task, requiring the sort of knowledge of custom and tradition that ethnographers or anthropologists possess. *Pasta by Design* takes on the challenge of classifying these pasta types in a new way. Thanks to shared formal features, each shape exhibits a certain degree of mathematical relatedness, which determines its relative lineage and position in a conceptual family tree. Equally inspired by the science of *phylogeny* (the study of relatedness among groups of natural forms), we pared down the startling variety of pasta to 92 unique types, divided according to their morphological features. In a mathematical sense, the criteria of differentiation are 'topological invariants' involving the edges and

Fig. 4 Close up of Tortellini. *Pasta By Design* Thames and Hudson, London 2011

surface of each sample (such simple properties are common to widely different shapes and help understand accidental differences).

The layout of Pasta by Design is constant. On each page, a brief text explains the etymological and historical derivation of the pasta, as well as giving example dishes. Each member of the family is illustrated by its parametric equations, a 3D diagram and a specially commissioned photograph (Figs. 6 and 7).[4] Combined, these representations capture the essence of each pasta, and give a concise, elegant and unique expression of a familiar thing. The book concludes with the *Pasta Family Reunion*, a reshuffle of the Pasta Family Tree into a 'bubble diagram' (a seating plan-cum-map) in which various relatives of the extended pasta family are hoarded together by similarities in shape and feature.

The three parametric equations of each shape form the central part of the book. Named after pi Π, theta Θ and kappa K, three letters from the Greek alphabet,[5]

[4] By Venice-based artist and photographer Stefano Graziani, co-author of the book.

[5] Our original choice of characters Π Θ Γ—the first three consonants of the modern Greek spelling of *Pythagoras*—was dropped because Γ is already reserved. We settled for Π Θ K—the first three consonants of the modern Greek spelling for *monkey*.

Fig. 5 Close up of Saccottini. *Pasta By Design* Thames and Hudson, London 2011

these three equations form the ‘genetic code’ of the pasta shape (because the shape is 3D, we need three equations to represent it). Each equation sets the measurements in a separate dimension: width, length and depth. These equations feature simple mathematical functions that use the ranges defined above to ‘plot’ the form. Solving equations and for $j = 0$, $j = 1$, and $i = 0$ to 25 on page 25 for instance, gives us the circular cross-section of *gramigna*. We used this and other, similar results to place the shape in the Pasta Family Tree.

Conclusion

Can this mathematical approach produce better pasta? Absolutely. As the Family Tree of *Pasta by Design* attests, analytic geometry will do for pasta what it does best: help us make sense of the endless variety of the sensible world by providing a unified

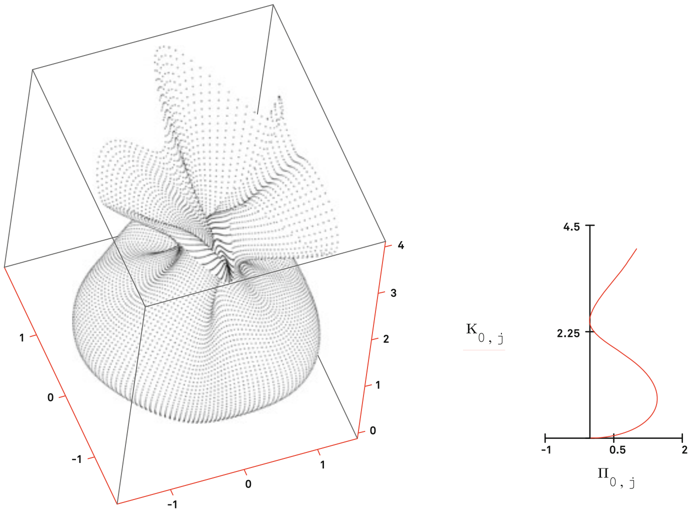

Fig. 6 Parametric equations of Saccottini. *Pasta by design* Thames and Hudson, London 2011

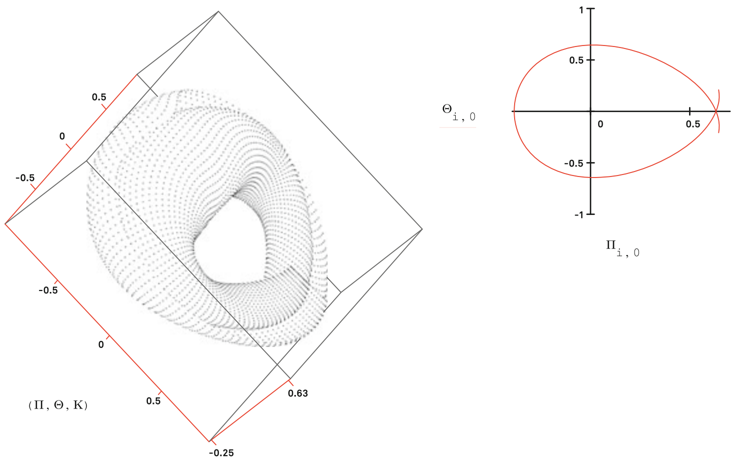

Fig. 7 Parametric equations of Tortellini. *Pasta by design* Thames and Hudson, London 2011

model of representation and understanding. Besides, the topological properties of each shape (extrusion, crenellation, bundling, and shearing) have a direct impact on how it absorbs heat and water, cooks, soaks in liquids, and retains a sauce—a shape may even be specifically formed for a given sauce. Such properties may be deployed to measure—as they have been for centuries—being empirically well-known to pasta makers of the past. In terms of manufacturing, there is still some way to go before enjoying the benefits of a mathematical approach. Any effort to design a new shape devised exclusively with periodic equations would have to contend

with the limitations of manufacturing by extrusion, which handles complex double curvature with unparalleled ease, while struggling to reproduce a simple right angle (or discontinuity, in geometric terms).

The world of mathematics, with its complexity and arcane symbols, can be forbidding at the best of times—but not so with this book. In the course of planning it, we encountered time and again the warmest of reactions. On receiving a sample page of the unsolicited typescript, our (future) editor at Thames & Hudson saluted, enthusiastically, our ‘insanity’. A prominent structural engineer, inspired by the sight of pasta shapes lying around our office, sighed and wished he too had more time to ‘have fun’. A high school student on the lookout for a placement had his headmaster call us about the ‘job with the pasta’ (he sounded unsure). In a coffee shop near central London, a stranger with impeccable manners interrupted our conversation to enquire about the proofs on the table. And, finally, during an international mathematics conference Amy Dahan, the distinguished historian of science, leaned towards me to inquire what IJP project associate Jean-Aimé Shu, with whom I produced this book, was ‘going on about’ at the lectern. I could see why. His talk on topological invariants was broadcast on a video screen filled, by way of a surreal disclaimer, with a nine-foot-tall photograph of a golden *saccotino*. She seemed genuinely intrigued.

The Demolition of the Church of San Geminiano and the Perception of Piazza San Marco in Venice

Alberto Giuliani

Until 1807, when it was pulled down, the church of San Geminiano in Piazza San Marco acted for centuries as the centre of visual attraction of the western side of the Piazza. The church served, at the same time, as a link between the Procuratie Vecchie and Nuove which surrounded it from both sides (Fig. 1). Today, the Piazza San Marco appears as in the scheme in Fig. 2, where the hatching indicates the position of the demolished church. The story of this church was tormented, having undergone several rebuildings including a complete shift from its original position in twelfth century.

Structural Vicissitudes and Sansovino's Project

A first church centrally planned was dedicated to saints Geminiano and Mena before the ninth century; it was located in a garden that originally occupied half of the actual Piazza, behind the canal (rio) Batario which run across the place. Under the rule of Sebastiano Ziani (Doge in 1172–1178), the entire area of the Piazza underwent radical changes: the first building of the Basilica di San Marco erected under the Domenico Contarini (1042–1071) was enlarged, the quay corresponding to the actual Piazzetta was realised, and the two ancient columns named after saints Marco and Todaro were placed.[1] In this period the first building of the Procuratie Vecchie was begun; the rio Batario was filled in, enlarging the area facing the Basilica. The first building of San Geminiano was then demolished, and the church was immediately reconstructed across the new western limit of the Piazza. The structure of the new church followed the previous central plan that Byzantine

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Many warm thanks to Isabella Cecchini and Claudio Bonechi for their help. The usual disclaimers apply.

[1] M. Agazzi, *Platea Sancti Marci. I luoghi marciani dall'XI al XIII secolo e la formazione della piazza*, Venezia 1991; J. Schulz, *La piazza medievale di San Marco*, in "Annali di architettura", 4–5 (1992–1993), pp. 134–156.

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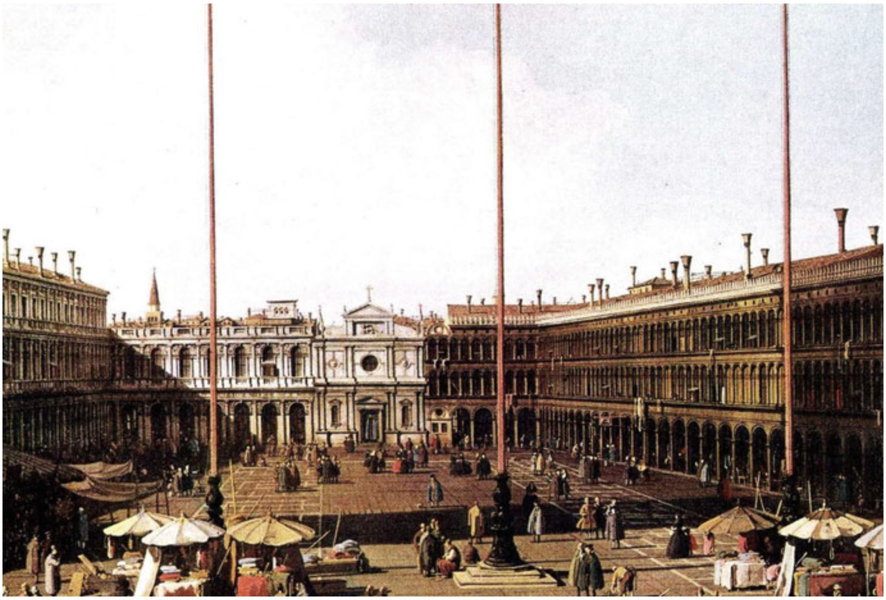

Fig. 1 Canaletto, *Piazza San Marco looking toward San Geminiano*, 1735 c. Oil on canvas (Roma, Galleria Nazionale d'Arte antica)

architectures had inspired. In his first edition of the map of Venice in 1500 Jacopo de' Barbari foreshortened the Gothic brick façade of San Geminiano with its three-spires profile and its lateral bell tower (Fig. 3).

The Doge Leonardo Loredan (1501–1521) commissioned in 1505 a local architect, Girolamo dal Legname, to rebuild the church following the existing spatial framework. In 1557 the works, still unfinished, were assigned to Jacopo Sansovino who completed the interior and realized a new façade (Fig. 4); in 1581 Sansovino's son, Francesco, a renowned writer, described the new church in his guide to Venice (*Venetia città nobilissima et singolare*) as "possibly the most adorned than any other church in city: being inside and outside encrusted with marbles and Istrian stone, it's therefore extremely lavish".[2] Jacopo Sansovino and his family were buried inside, at the Cappella del Crocefisso; San Geminiano also hosted the graves of all the choirmasters who worked in the Basilica di San Marco.[3]

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[2] Author's translation: "forse la più ornata di qualsivoglia altra nella città: perciò che essendo di dentro et di fuori incrostata di marmi , et di pietra istriana, è ricchissima". F. Sansovino, G. Martinioni, *Venetia città nobilissima et singolare, descritta in XIIII libri da M. Francesco Sansovino . . . Con aggiunta di tutte le cose notabili . . .*, Venezia 1663, p. 109.

[3] E. Bassi, *Tracce di chiese veneziane distrutte. Ricostruzioni dai disegni di Antonio Visentini*, Venezia 1997, pp. 25–27.

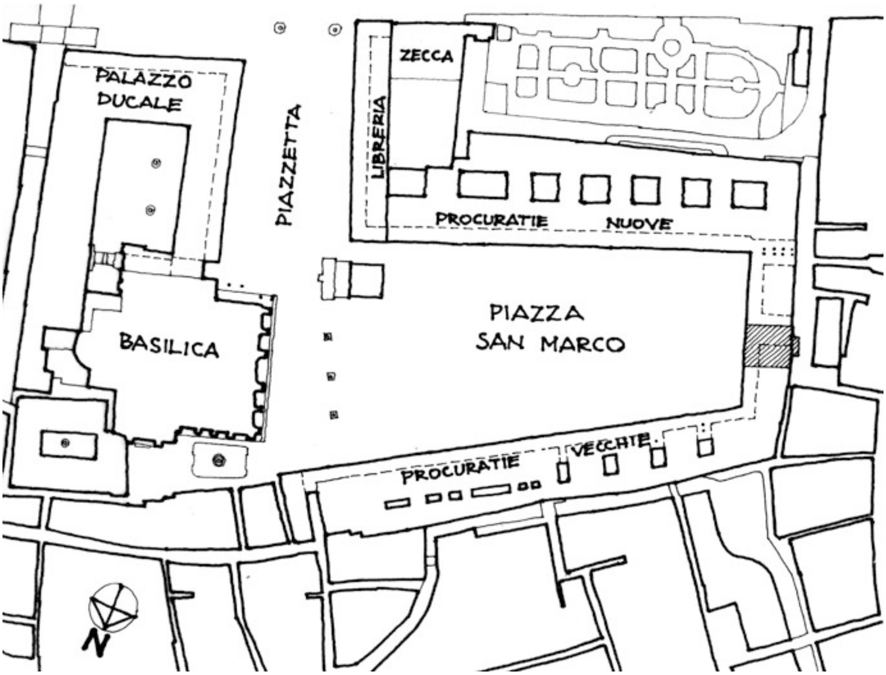

Fig. 2 Planimetry of the Piazza (drawing A. Giuliani)

The church occupied a precise role in the complex system of signs and symbols forming the Venetian civic ritual: every year on Low Sunday the Doge, the priests serving at San Marco, the members of the government, and the foreign ambassadors walked in procession to San Geminiano to hear Mass and then come back to the Basilica. The ceremony was meant to symbolize a penance imposed on the Doge by the priests of San Geminiano for his ancient predecessor, Sebastiano Ziani, had demolished the first building. The procession back to the Basilica di San Marco reached its peak when the participants paused where the ancient church stood: here the Doge in office made solemn amends for the destroying action that Ziani committed.[4] This ritual took incessantly place until the end of the Venetian republic in 1797.

Besides the role of being a plastic node and a hinge for the western side of the Piazza,[5] the church that Sansovino planned[6] evoked, from the linguistic point

[4] Sansovino-Martinioni, *Venetia* ..., pp. 109, 496–497.

[5] G. Romanelli, *Il Forum publicum e le "cerniere" architettoniche del Sansovino*, in "Architettura e utopia nella Venezia del Cinquecento", L. Puppi ed., Milano 1980, pp. 89–92.

[6] On the architecture of the church and its relations with the Piazza see R. Gallo, *Contributi su Jacopo Sansovino*, in "Saggi e memorie di storia dell'arte", I (1957), pp. 96–100; M. Tafuri, *Sansovino e l'architettura del '500 a Venezia*, Padova 1969, p. 145; D. Howard, *Jacopo Sansovino*, New Haven/London 1987, pp. 77–87; M. Morresi, *Jacopo Sansovino*, Milano 2000, pp. 336–340.

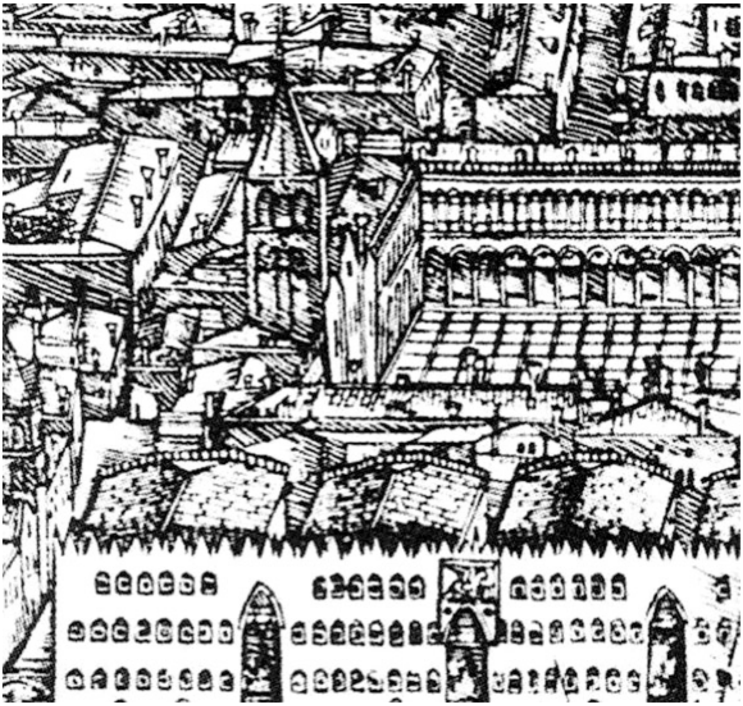

Fig. 3 Jacopo de' Barbari, detail of the *Veduta prospettica di Venezia*, 1500. Woodcut (Venezia, Museo Correr)

of view, the memory of the rebuildings that become an opportunity to a complex symbolic action. Here the themes of the rebirth and of the morphological roots take shape through explicit figurative recalls to Venetian tradition, like the two small old-fashioned bell towers in the crowning, or the austere sobriety in the architectural frame almost resembling a Venetian early Renaissance origin. With respect to the architectural ethic applied by Sansovino, however, such recalls cannot be imagined as elements of repertoire to be used *tout court* in every new religious building in Venice, but only on the occasion of planning the only church that had the fortune of facing the *capella ducis* (the Basilica) containing the relics of the city's patron saint.

Along the creative arc of Jacopo Sansovino the façade of San Geminiano represents a kind of *unicum*. Here, through a temperate classicism, the architect dialectically summarizes symbolic needs related to the function of the church as qualified counterpart of the Basilica, bearing also the figurative echoes of the previous buildings. On the other hand, more advanced compositional experiments may spread out freely, in the very same years, and without such demanding comparisons. The realization of the façade of San Giuliano, a church that overlooks

Fig. 4 Luca Carlevarijs, *Chiesa di S. Giminiano*, 1703. Etching (private collection)

the commercial street of the Mercerie very near to San Marco, planned by Sansovino and strictly correlated to San Geminiano, offers an obvious proof.

Inside the corpus of literature on Sansovino any serious attempt to reconstruct the original aspect of the church collides with the contradictory multiplicity of iconographical sources. In fact, any of the various paintings and prints showing this side of the Piazza, from Canaletto to Guardi and Marieschi, reproduce a different façade of San Geminiano. This problem favoured a number of scholarly contributions that used often uncritically the visual material which the historiographical tradition cherished. A leading scientific reliability has been continuously granted, in particular, to nineteenth-century architectural surveys, and especially to those contained in an engraved book on the best palaces and monuments in Venice (*Le fabbriche e i monumenti cospicui di Venezia*), published in 1815 by the hands of Leopoldo Cicognara, Antonio Diedo, and Giannantonio Selva, three main protagonists of Venetian cultural and artistic life of the time; however, most of these surveys were drew up after the demolitions carried out in Venice under the Napoleonic rule.[7] The disappearance of the church, as in the case of Santo Spirito

[7] As far as we know, among the architectural drawings only Antonio Visentini in his *Admiranda urbis venetae*, III, 22, described the church before its demolition.

in Isola, probably allowed its inescapable critical marginalization, but was not able to wipe out its essential role in shaping the image of the Piazza.

Andrea Tirali's New Paving and the Problem of Perspective Axes

After the Procuratie Nuove were erected by Vincenzo Scamozzi in 1582, and completed by Baldassarre Longhena in 1640, the new eighteenth-century paving represents a further important stage in completing and defining the morphology of the Piazza.

In 1723 the Procuratori di San Marco, in charge of any work on the site, commissioned the *proto* (chief surveyor) Andrea Tirali to build a "new paving following the old plan, which the inhabitants of this city and mostly the foreigners had admired throughout the centuries".[8] Before this date, other interventions are known. The first documented paving of the Piazza, with a burnt brick herringbone pattern, went back to 1266 under the Doge Ranieri Zeno; in 1392 the bricks were replaced and mounted into geometric partitions that were made with light stone stripes, the so-called *quadreti*.[9] Other periodic works of maintenance and replacement of the run-down portions were carried out along seventeenth and eighteenth centuries; the paving was also extended to the zone of the Piazza that the architect Vincenzo Scamozzi freed for building the Procuratie Nuove.[10]

The condition existing immediately before Tirali's works can be appreciated through some iconographical texts, and first of all through Luca Carlevarijs' engravings, which would have become prototypes of numberless replicas.[11] In his

8 Author's translation: the "nuovo salizo secondo il vecchio comparto, che da secoli è stato ammirato dagli abitanti di questa città e maggiormente dagli esteri". On the facts and significates of the eighteenth-century paving see L. Puppi, *La gondola del procuratore. Committenza e peripezie di collezione di quattro dipinti di Canaletto*, in «Bollettino dei Civici Musei Veneziani», (1983–1984); G. Bellavitis, *La proiezione mondana del sacro sull'area marciana e la questione dei tre stendardi nel primo Settecento*, in *Scienza e tecnica del restauro della basilica di San Marco. Atti del convegno internazionale di studi* (Venezia, 16–19 maggio 1995), A. Lepschy and E. Vio eds., Venezia 1999, pp. 227–253.

9 G. Saccardo, *Appendice alla Relazione degli scavi in Piazza S. Marco,* Venezia 1892, p. 40.

10 Demolitions involved both the Ospizio Orseolo and offices and houses related to the Procuratori; see M. Agazzi, *Platea Sancti Marci*. On the contemporary debate on Scamozzi's interventions see M. Tafuri, *Venezia e il Rinascimento*, Torino 1985, pp. 252–271.

11 *Le fabriche, e vedute di Venetia disegnate, poste in prospettiva et intagliate da Luca Carlevarijs*, Venezia 1703. These views had an enormous success and they were repeated and republished, often without mentioning the original, by several Italian and European editors. See *Luca Carlevarijs. Le Fabbriche e Vedute di Venetia*, I. Reale ed., Venezia 1996.

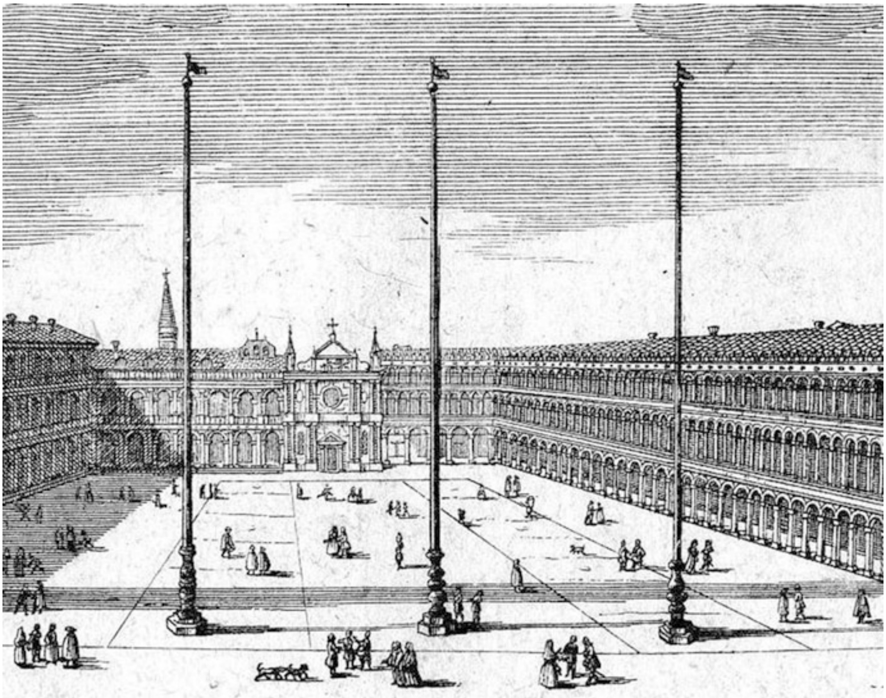

Fig. 5 Ludovico Ughi after Luca Carlevarijs, detail from the *Pianta topografica della città di Venezia*, 1729. Etching (private collection)

view of the Piazza towards San Geminiano, the strips of the old plan are clearly oriented to frame Sansovino's façade in perspective, starting from the end of the seventh arch of the Procuratie Nuove (Fig. 5). This fact seems confirmed from other iconographical sources dating from the early eighteenth century, as one engraving by Bernard Vogel from Hans Richter shows (Fig. 6).

Tirali too, in his sophisticated plan, chooses San Geminiano façade as one of the elements that generate the axes of the new paving as it is clearly testified in the engravings that celebrated the new design and published by the Zucchi brothers (Figs. 7 and 8). In particular, the line joining the axis of the church with the central flagpole seems to become the governing element through which the parallel lines of the central compartment and of the Greek fret sides are drawn. The new paving, successfully mediating between the conflicting geometrical lines of the Piazza, is built with grey trachyte slabs and decorations in white Istrian stone (Fig. 9). However, the detailed analysis of the old design in the Zucchi planimetry shows that the central field was markedly shifted towards the Procuratie Nuove (Fig. 10).

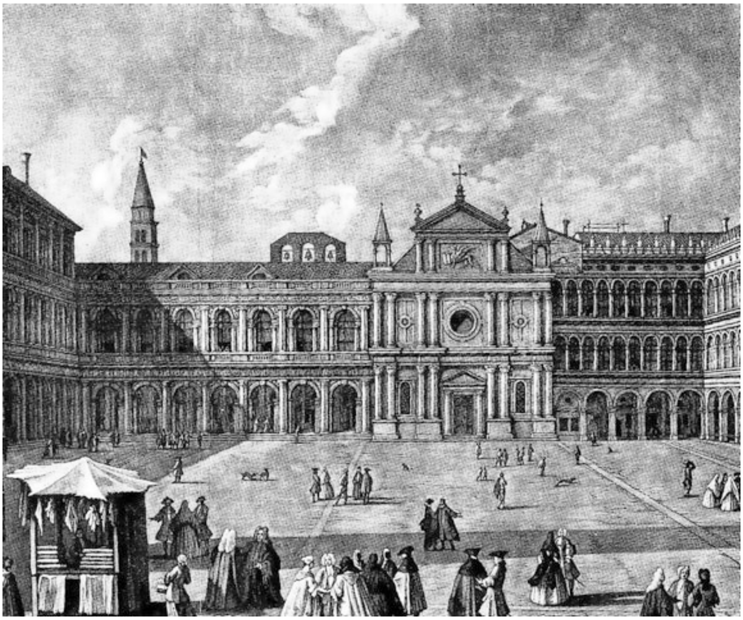

Fig. 6 Bernard Vogel after Johann Richter, *Veduta della Piazza di S. Marco verso la Chiesa di S. Geminiano*, detail, 1735 c. Etching (Venezia, Biblioteca del Museo Correr)

In comparison with reality, this explicit and probably emphasized diversity, as it is testified in etchings and paintings (Figs. 4, 5 and 6), could hypothetically be ascribed to Tirali's intent to give wider merits to his future works, and firstly to amend the imprecision of the old paving. It is well known how the first draft of his project, which initially moved the three flagstaffs, come across some important changes during the works, investing both the baroque inflection of the headpieces in the lateral decorative patterns and the oval elements of the Greek frets. It is unlikely, however, that the general layout—that is, the projective synthesis from which the geometry of relations between the paving and the church springs—had changed.

The new paving was realized between February 1723 and August 1734. In terms of planning philosophy, Tirali had to face the symbolic and visual link between the *two* churches connoting the opposite sides of the Piazza, and to emphasize it.[12] The stone inlay evokes the ritual way from the Basilica to San Geminiano, a sort of processional carpet regulating visually in the Piazza the perception of space that was freely usable. However, it was a theoretic, visual way. As we know from the

[12] C. de Brosses, *Viaggio in Italia*, Roma-Bari 1973, p.106: the Piazza is "regular, square, long, limited at its two extremities from the churches of San Marco and San Geminiano, and on its sides from the old and the new Procuratie" ("è regolare squadrata, lunga, limitata alle due estremità dalle chiese di San Marco e San Geminiano, e sui lati, dalle Procuratie vecchie e nuove"). Author's translation De Brosses visited Venice in 1739.

Fig. 7 Fratelli Zucchi, *Veduta della Piazza di S. Marco con il disegno del novo salizo*, c. 1724. Etching (Venezia, Biblioteca del Museo Correr)

iconography of the ceremonies that took place in the Piazza, processions wound along a complex route, and rarely, it seems, along a straight way—with the possible exception of the visit (*Andata*) of the Doge at San Geminiano.[13] In confirmation of this hypothesis, we might observe that, after all, Tirali's main task was to offer a convincing planning answer to the necessity of conferring dignity, according to the rules of art, to a *void*, practicable space of the Piazza. To this end it was permitted to set aside any established processional itinerary to orchestrate freely the paving inlays. The layout on the ground had also the task of ruling other and more pragmatic functions.

The theatre of the Piazza traditionally was a market and commercial place, where Venetian republic, together with the symbols of its political and religious power, proudly displayed its merchant origins. During the week of Ascension Day a renowned fair was held every year in late spring; here a great number of shopkeepers and artisans, Venetian and foreigners, flocked to sell their merchandise in temporary stalls, whose position was ruled through a meticulous assignment of places under the supervision of the Procuratori di San Marco. A drawing assigned to late sixteenth century displays in detail the dislocation of the stalls and the rows of trades (Fig. 11). The path called *Liston delli Oresi* (Goldmiths' row) lines up on

[13] On the processions in Piazza San Marco see E. Muir, *Il rituale civico nella Venezia del Rinascimento*, Roma 1984; L. Urban, *La Piazza: cerimonie e feste*, in *Le procuratie nuove in Piazza San Marco*, Roma 1994, pp. 223–270.

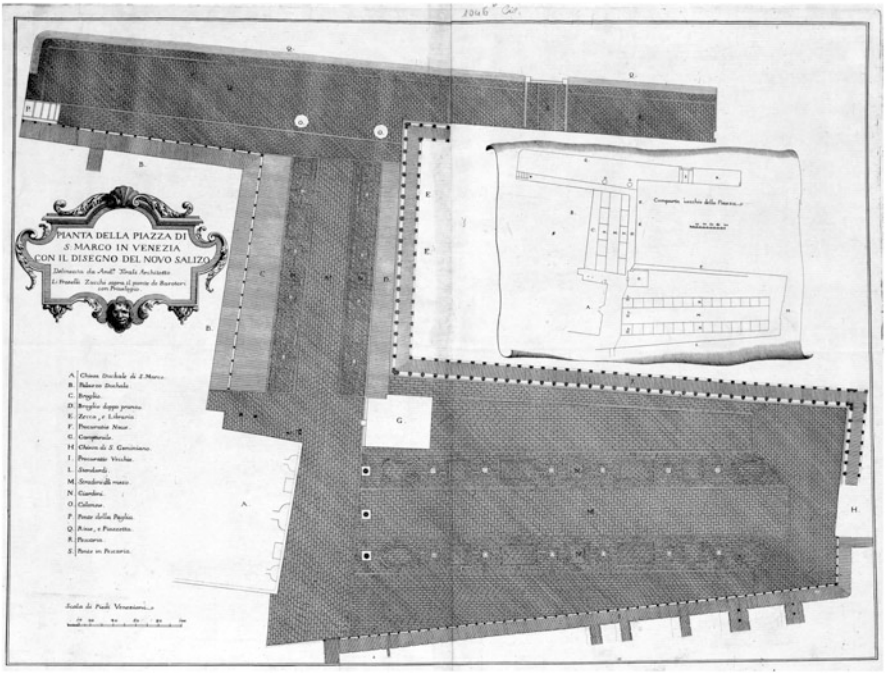

Fig. 8 Fratelli Zucchi, *Veduta della Piazza di S. Marco con il disegno del novo salizo delineata da Andrea Tirali Architetto*, plan, c. 1724. Etching (Venezia, Biblioteca del Museo Correr)

both sides 24 stalls and anticipates the central field of the *new* paving planned by Andrea Tirali, and it probably coincides to the central field of the *old* paving that still existed in the Piazza when the drawing was made. The line going from San Marco to San Geminiano, then, was also the main axis of fairs, as clearly shows one painting by Gabriel Bella (1730–1799) with the "old" Ascension Fair, *La fiera della Sensa Vecchia* (Fondazione Querini Stampalia, Venice). The stripes outlined by the white Istrian stone become as well the symbol and the permanent sign of interlaced periodical activities that took place here, scanning times and rites in the life of the republic and substantiating its myth: from religious processions, to public celebrations, to merchant fairs. Far from being a simple element of street furniture, the paving (*salizo*) becomes a synthesis of history and memory; the juxtaposition of inscriptions inside the bright stone stripes, that were traditionally inserted *ab antiquo* to mark the fair stalls, mingles with the foundation myths through the memory of the ancient church of San Geminiano. What Tirali imagined, though, was the cause of a few collateral problems echoing in the Piazza; one of them was the extreme proximity of the upper right-hand edge to the last arches of the Procuratie Vecchie. The architect probably thought to get out of it designing the terminal volutes that were later cancelled, and that were intended to taper elegantly the paving design towards San Geminiano (Figs. 9 and 12). Moreover, the limit of the central field in

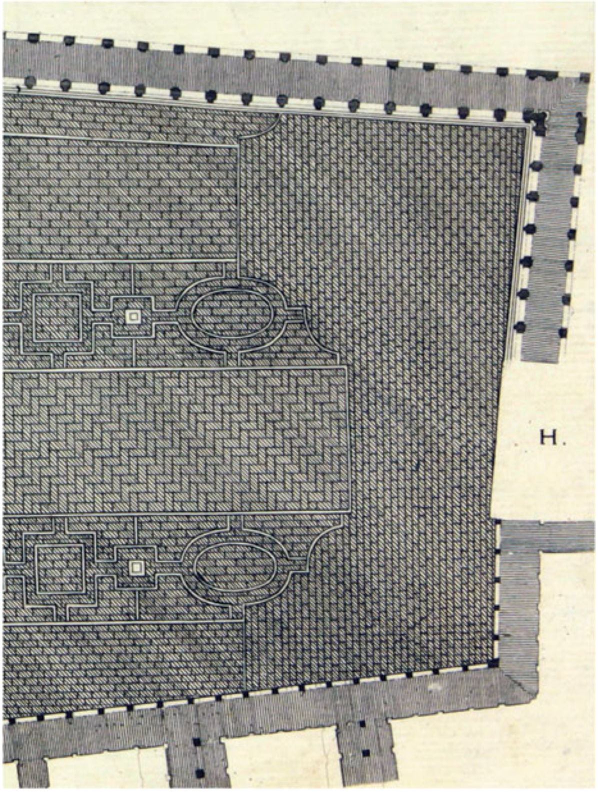

Fig. 9 Fratelli Zucchi, *Veduta della Piazza di S. Marco, nuovo comparto,*detail. Etching (Venezia, Biblioteca del Museo Correr)

the paving was pulled back with respect to the façade, as to identify a wider area, a sort of parvis pertaining to the church as it happened to the space on the opposite side that the flagpoles in front of the Basilica defined.[14]

As it is well known, any visual document drawn once the works were completed shows different positions for the bright stone stripes of the paving, usually depending on the chosen visual angle or on the changeable positioning of the three flagpoles. If the uncertain scientific reliability of painters is by now widely proved,[15] to evaluate the architectural reliefs after the paving was completed creates more complex and insidious problems. In his detailed plan of the Piazza (1777) Pietro Soardi shifts the central stripes towards the Procuratie Nuove (Fig. 13). In 1776 the architect Bernardino Maccaruzzi, in a preliminary project for a new set of wooden stalls

[14] G. Bellavitis, *La proiezione mondana del sacro*, cit., pp. 230–242.

[15] A. Corboz , *Canaletto. Una Venezia immaginaria*, Milano 1985.

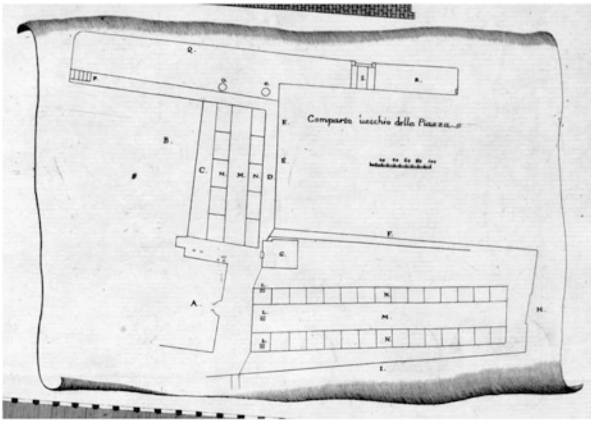

Fig. 10 Fratelli Zucchi, *Veduta della Piazza di S. Marco, vecchio comparto,* detail. Etching (Venezia, Biblioteca del Museo Correr)

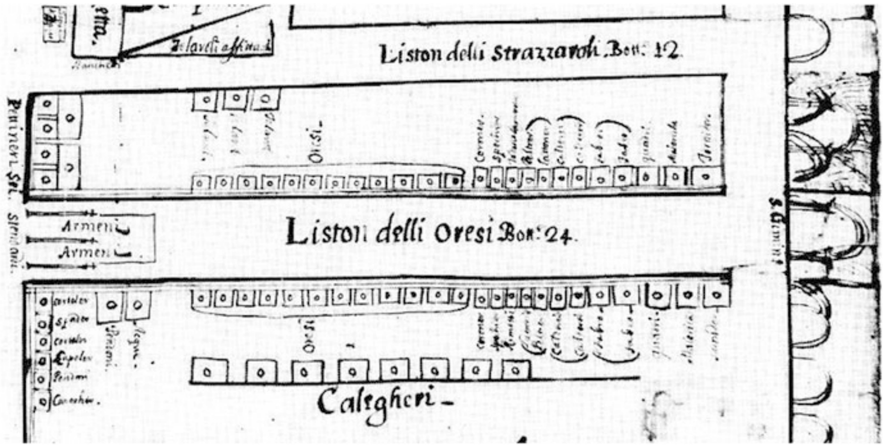

Fig. 11 Anonymous, *Woodstalls for the Ascension Fair*, late 16th century (attributed). Drawing (Venezia, Archivio di Stato)

for the Ascension fair,[16] lays down the plan parallel to the line joining the axis of San Geminiano façade with the central flagpole (though this is not indicated in the plan) (Fig. 14). Following such an alignment means to sacrifice, on the Procuratie Vecchie side, many of the outer stalls that Maccaruzzi variously cut out to leave the indispensable outer passage. The planimetric planning of this project is particularly considerable, since the stalls were intended to be assembled every

[16] L. Urban Padoan, *La festa della Sensa nelle arti e nell'iconografia*, in "Studi Veneziani", 10 (1968), pp. 291–353.

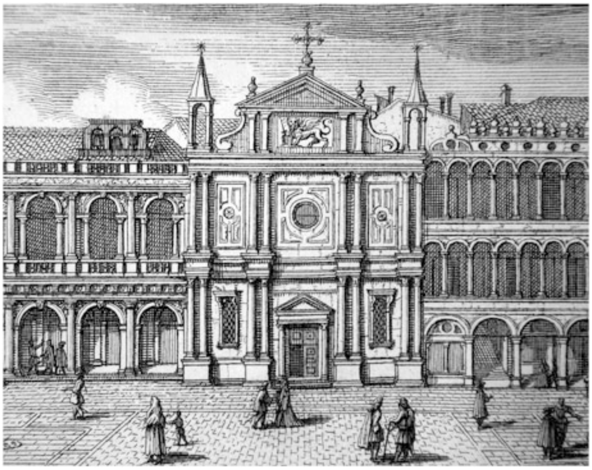

Fig. 12 Francesco Zucchi, *Chiesa di San Geminiano*, from *Teatro delle fabbriche più cospicue . . .* , 1740. Etching (Venezia, Biblioteca della Fondazione Querini Stampalia)

year during the Ascension fair and hence to illustrate the symbolic and perceptive priorities of any intervention to be realized in the Piazza. The logic in this specific plan could not be too different from the one Tirali adopted in his paving plan; the latter had a permanent feature and hence represented a kind of model for the correct interpretation of the *symbolic axis* of the Piazza. On occasions, instead, as in the case of the ceremonies offered to the grand duke of Russia in 1782 setting up in the Piazza an arena designed by Antonio Codognato, a *functional axis* could be preferred. Here some scenographic expedients were necessary to allow the confident operation of using the space of the Piazza at its maximum: the Sansovinian church would have been totally cancelled from sight through a background pavillion. In the celebration's plan the projection of the first paving band falls at the end of the seventh arch, while the other band is shifted—though slightly—towards the inner part of the façade.

In the nineteenth century the inconsistency of visual informations seems even less understandable, facing more science and more accuracy in drawing up graphic documents. But an important consideration is now due: most planimetries are in fact carried out *after* 1807, when the church was pulled down. From this moment on, missing one of the pivotal elements for its formal and conceptual legitimacy (the façade), the ideational origin of the paving design begins to be even less obvious and less comprehensible.

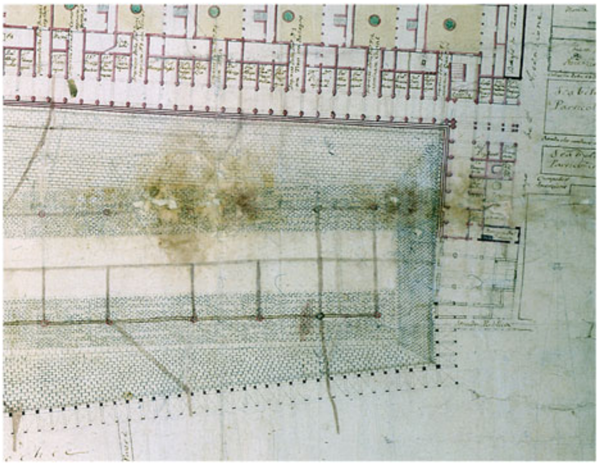

Fig. 13 Pietro Soardi, *Planimetria dell'area marciana*, 1777, detail. Ink on paper (Venezia, Biblioteca del Museo Correr)

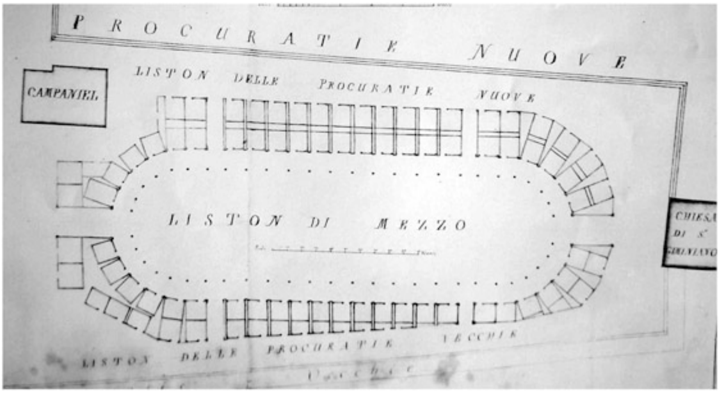

Fig. 14 Bernardino Maccaruzzi, *Progetto preliminare per la Fiera della Sensa Nuova*, 1776. Ink on paper (Venezia, Biblioteca del Museo Correr)

The Demolition of the Church and the Nineteenth-Century Transformations

Disputes and vivid debates accompanied the demolition of the Sansovinian church.[17] According to the architect Giannantonio Antolini, the new Royal Palace of the Napoleonic reign in Venice was initially expected to take place inside the Procuratie Nuove, that were fit to their new functions having received a new waterfront on the

[17] G. Romanelli, *Venezia Ottocento. L'architettura, l'urbanistica,* Venezia 1988, pp. 75–97

Bacino after the pulling down of the old Granai di Terranova (the official granary during the Venetian republic, visible in the foreground in Fig. 3). The great staircase to the royal apartments would have coincided with the staircase of the Sansovinian Libreria Marciana, and the waterfront would have been adapted building blind arches reiterating the front of the Zecca. This solution, though less invasive, gave rise to doubts, expressed by the new political representants. The façade of the Procuratie Nuove on the Piazza remained substantially unchanged and the representation of the new imperial power gave rise to a project that was considered exceedingly marginal. The hypothesis of building new spaces in the western side of the Piazza, with the pulling down of San Gemininano, then gained height.

In an atmosphere of harsh controversy several proposals flourished, even from amateurs. It is worthed to mention the project proposed by Gaetano Pinali (Fig. 15), who suggested the realization of a pronaos in a gigantic Corinthian order to mediate between the old and new Procuratie, which were to remain nearly intact. From a technical point this project was afflicted with several planning mistakes both regarding the great staircase of honour (facing the Piazza first, then built longitudinally) and the alignment of the new building. Moreover, the decision to demolish or not the seven surviving arches of the Procuratie Nuove adjacent to San Geminiano was intensely discussed. The question was settled maintaining these arches and pulling down those pertaining instead to the Procuratie Vecchie, since they had a divergent angle from the new front of the Piazza. Antolini was relieved of his position and Giuseppe Maria Soli was appointed to draw up the final version of the project: the arches of the Procuratie Nuove were to be extended on the entire front until they meet the angle with the Procuratie Vecchie; the new building would have been surmounted by a wide attic in bas-relief tentatively mediating the divergent heights of the two long sides of the Piazza (Fig. 16). The building was completed afterwards by Lorenzo Santi, who planned also the front towards Calle dell'Ascensione.

Subsequent planimetries record accurately the new state of the site, but they are much more approximate in placing correctly the eighteenth-century paving. Since the church façade is not visible anymore, limited rotations or translations of the paving lines could be drawn without invalidating the representation of the Piazza. The complex planning equilibrium reached by Tirali had lost most of its comprehensibility and hence of its symbolic and architectural meaning.

Once closed the century-old process of completion and definition of the Piazza, it is only towards the end of the nineteenth century that a sort of routine and last intervention takes place. However, we will see how this intervention will have important consequences in the perception of the Piazza.

Let's turn to the complete reconstruction of the eighteenth-century paving, that was then in advanced decay, in the 1880s.

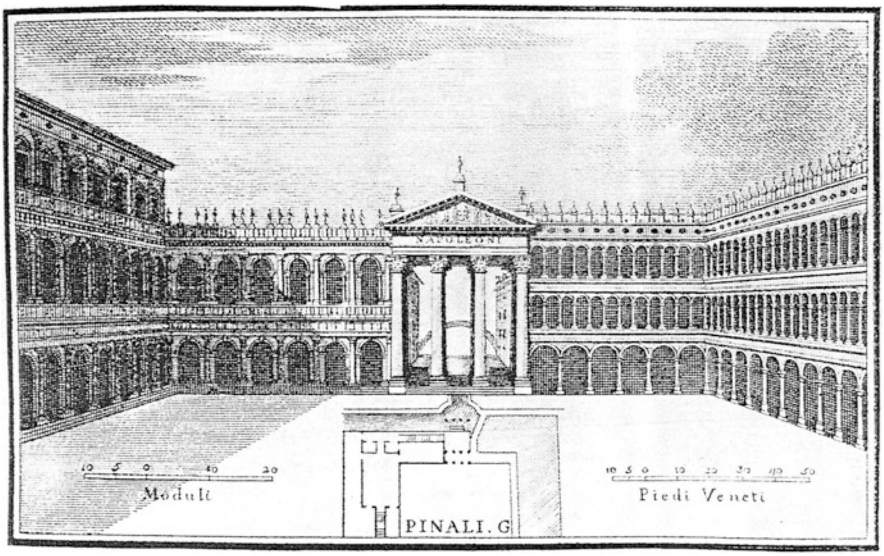

Fig. 15 Gaetano Pinali, *Progetto per la sistemazione del Palazzo Reale nelle Procuratie Nuove. Fronte su piazza S. Marco e pianta*, 1808. Etching (Venezia, Biblioteca Museo Correr)

Fig. 16 Aerial view of Piazza San Marco, Ala Napoleonica. Current situation. Photograph

The replacement of the paving has been a main theme throughout the nineteenth century.[18] But only in 1883 the city council approved a project to remake the entire paving in the Piazza and in the Piazzetta, after having completely removed the old stone slabs.[19] The minutes of a meeting on 18 July 1883 report how it was by then considered “unavoidable to remove the artistic flagpoles, that will be placed in the same site having tried to laid down their bases matching the partitions of the frets, whose design is scrupulously maintained, adjusting only some symmetry defects.”[20] In short, the city engineers, while interpreting operatively such ambiguous prescriptions, could have taken the liberty of changing both flagpoles' position and frets' form and position.

The precise survey of the subsoil of the Piazza on this occasion provided important informations on the foundations of the ancient buildings and on the exact site of rio Batario;[21] the map marks with a hatching the design of the old paving that was being dismantled. It is represented bearing the same decoration that can be seen in the plan prepared to celebrate the Russian grand duke (1782). Interestingly, stripes are sketched as in the original project by Tirali, that is symmetrically oriented to the axis of the lost façade starting from the seventh arch.

The Perception of the Piazza, the Alignments, and the Seventh Arch

Let's come to nowadays. If we try to extend the axes of Istrian stone bands defining the middle field of the paving, on the left-hand side we intercept half of the sixth arch of the Procuratie Nuove, not the seventh (see the engraving in Fig. 12). We suggest hypothetically that the entire paving decoration underwent a rotated translation towards the Procuratie Nuove in 1883; missing the church façade which acted as a guide, the intersection point of the second left-hand longitudinal band with the western side of the Piazza has been shifted of nearly 1.50 m (Fig. 17). This result probably depended on the adjustment of the “symmetry defects” that led also to a vaguely trapezoidal form in the central course, making its contours converge towards the west side of the Piazza. Even the sequence in frets' design is shifted of one square module with double frame, changing the general rhytm.

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18 For the project's chronology see Archivio Storico Comunale di Venezia, Atti del Consiglio Comunale di Venezia, July 18, 1883 (*Progetto di ricostruzione del pavimento della Piazza di san Marco, e conseguenti proposte*), p. 114.

19 For the works see L. Dalle Piane, *La pavimentazione di piazza S. Marco. Note per la conservazione e la manutenzione*, in “Restauro e tecniche”, 19 (19929, pp. 116–145).

20 Author's translation: “[è] inevitabile la rimozione degli artistici stendardi, i quali saranno riposti nello stesso sito avendosi cercato di mettere le piante delle basi in armonia con gli scomparti delle greche, il cui disegno viene scrupolosamente mantenuto, regolando soltanto alcuni difetti di simmetria.”

21 F. Berchet, G. Saccardo, *Relazione degli scavi in Piazza S. Marco,* Venezia 1892.

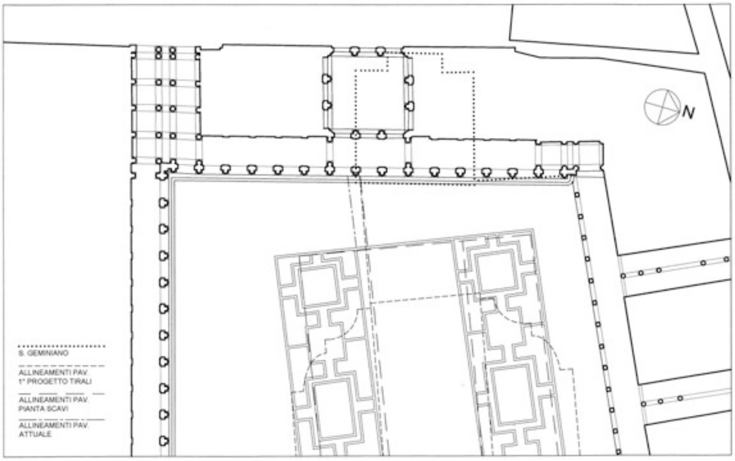

Fig. 17 Planimetry of the western side of the Piazza and outline of the paving. Drawing (A. Giuliani)

In the long course of planning and architectural definition of the Piazza, among the main elements that contributed to shape the *scene* (the Basilica, the bell tower, the Torre dell'Orologio, the Procuratie) even the church of San Geminiano has to be put in. In the dense game of mutual cross-references, the longitudinal axes of paving directed the view and concentrate it on the Sansovinian façade. The church then had a double function of *caesura* and of *measure* of the space; in particular, the very long line of annular arcade, that is the St Mark's forum, found its own completion in the plastic episode of San Geminiano's façade, which contributed to determine the range relations for the entire urban space. Presently one of the consequences of its absence is the difficulty in grasping, especially from some angles, the exact *scale* of the Piazza. Moreover, the view's vastness is optically enhanced and emphasized from the continuity of the Procuratie that turn on three sides: the perspective effect is multiplied by the trapezoidal planimetric course and tends to have the sensation of a deceptive, limited space. This effect proves particularly evident with artificial lighting in the night. The resulting sensation however, though extremely fascinating, it is not the same that the artists who took turns in the time concurred to define. The almost metaphysical effect of *infinity* we notice today is deeply different from the grandeur and the *finiteness* of the space representing the civil and religious centre of the city. It was San Geminiano to constitute the element that gave the *measure* to the Piazza, bringing the view back to a *human* scale, and stressing its artistically antirhetorical vocation. The façade in fact measured nearly 15.70 m length and 18 m high,[22] analogously with the nearby San Giuliano which was indeed restricted in the small wide stretch of the Mercerie.

[22] M. Agazzi, *Platea Sancti Marci,* cit. p. 123.

Summing up, to an exact comprehension of the planning choices, which the architects working in the Piazza made, an important point of reference, contributing to give the right sense and value to these choices, is lost: essentially it is one of the problem's terms to be missing.

Proposal for a Temporary Reconstruction

The reconstruction project that we are presenting was conceived for the recent bicentenary of the demolition of the church in 2007. It stands in the tradition of the sceneries for pageantry and, specifically, in ideal continuity with the staging made in 1980 by Giorgio Bellavitis recontructing San Geminiano façade, and celebrating the exhibition of sixteenth-century Venetian architecture (*Architettura e utopia nella Venezia del Cinquecento*). We happened to learn about it after our project was completed: although there were no images of this staging in the catalogue, we were able to obtain some newspaper pictures taken during the assembly phases of the work (Fig. 18).

The details of our proposal have been defined on the basis of a philological study of iconographical sources using both a chronological and historical comparison and a stylistic verification. In this context, however, it was considered more useful to highlight the main conceptual points of our research, leaving the elucidation of choices for the proposed reconstruction of the Sansovino's facade (Figs. 19, 20, 21, 22, and 23) for another occasion.

To translate our work at a more advanced stage, we are facing further, particularly insidious, problems. There are two options for the reconstruction of the facade along the western side of the Piazza: the first is to start from the seventh arcade choosing not to be in line with the central field of the paving (Fig. 22); the second option consists instead of placing the facade in line with the central field, thus making it start from the middle of the sixth arcade (Fig. 23). The first solution, adopted in 1980, would be visually in line with the works of painters, while the second would render anew the conceptual logic of the cobbled paving of Tirali (Figs. 7, 8, and 9). This issue is still open and might develop further research.

The operational approach which has joined several important historiographical researches on the urban morphology of some Italian cities between the 1960s and 1980s has been one of the hallmarks of the best scientific culture of recent decades. Subsequently, the generations of younger academics, in the face of increasingly specialized cultural and professional training, have often refrained from any intellectual suggestion that could only lead to the formulation of hypotheses concretely transformative of reality. On closer inspection, the apparent contradiction between the culture of conservation and that of the project, which is merged to create a deep awareness and respect of the documents and of the historical artifact, has often led to the most fertile and durable historical-critical insights of recent decades. This goes beyond the dose of visionary utopianism in some critical foundational

Fig. 18 Temporary reconstruction of San Geminiano for the exhibition *Architettura e Utopia nella Venezia del '500*, 1980. Photograph (Venezia, Archivio Fotoattualità)

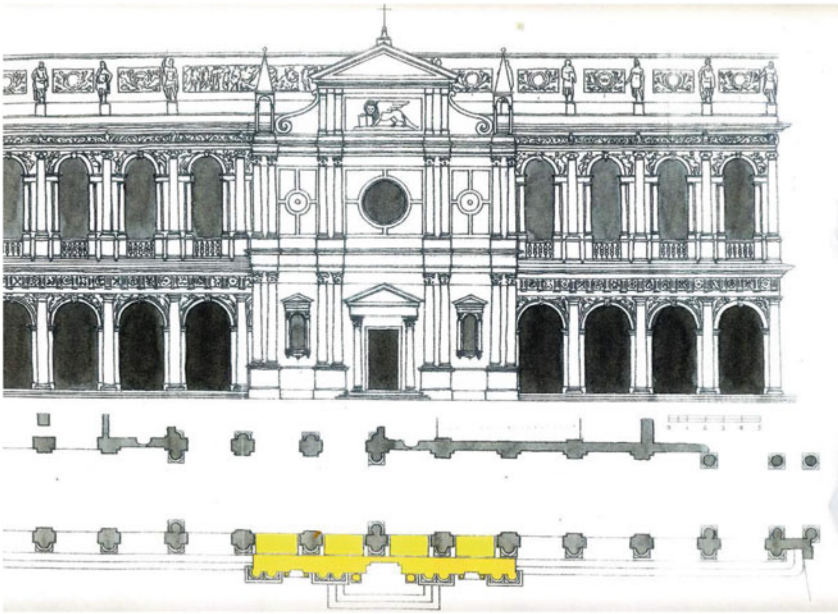

Fig. 19 Project of the façade. Drawing (A. Giuliani)

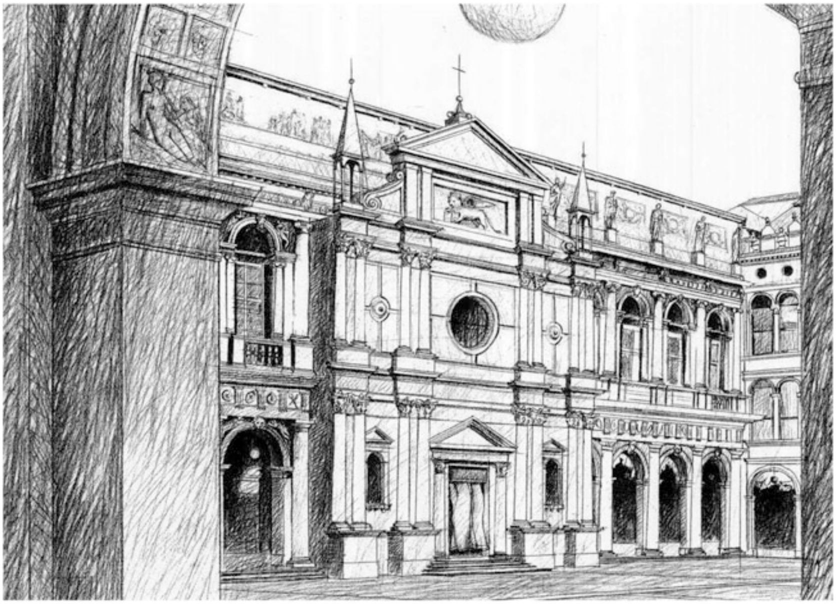

Fig. 20 View from the Procuratie Nuove. Drawing (A. Giuliani)

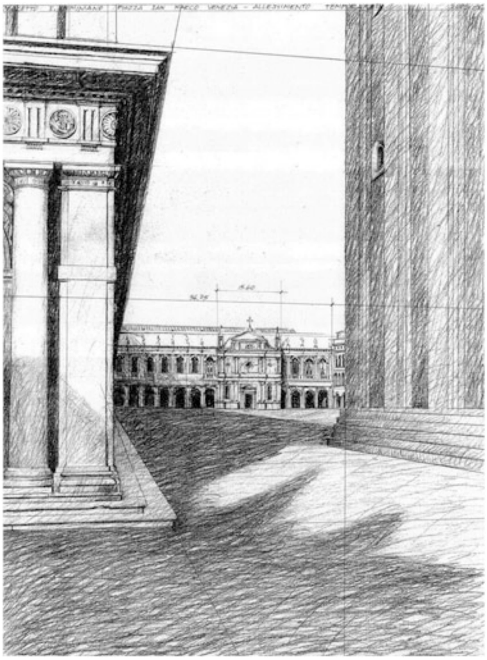

Fig. 21 View from the Libreria Marciana. Drawing (A. Giuliani; Berlin, Tchoban Foundation, Museum for Architectural Drawing)

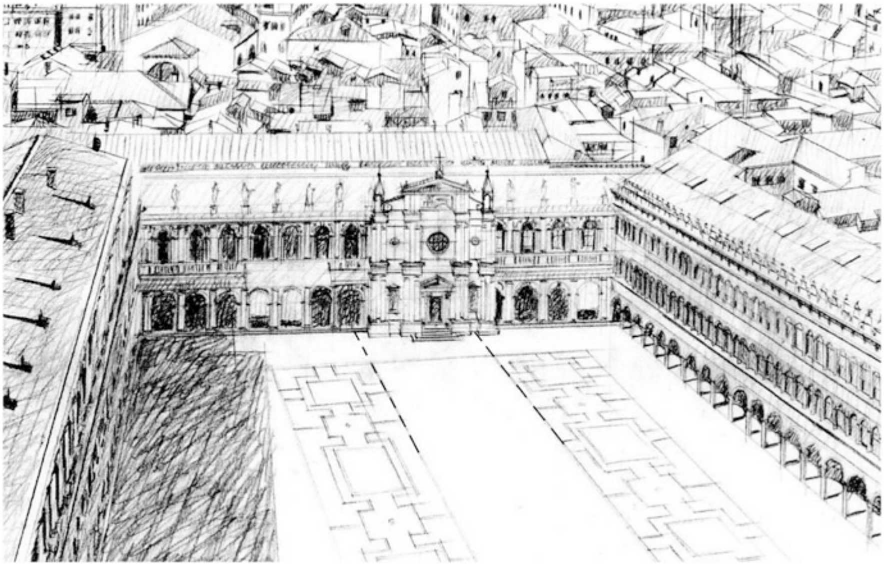

Fig. 22 View from the BellTower, first solution. Drawing (A. Giuliani)

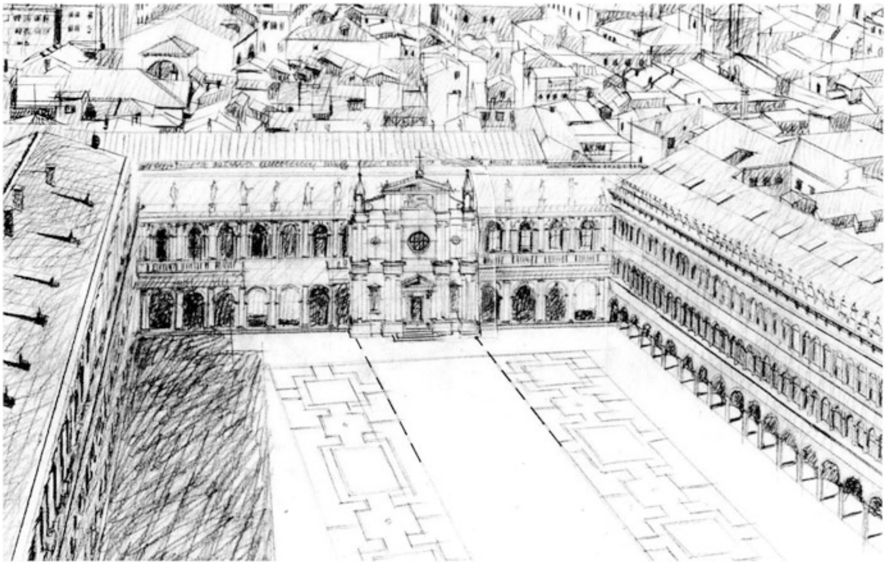

Fig. 23 View from the BellTower, second solution. Drawing (A. Giuliani)

studies that have helped to keep alive the design tension necessary for a true and effective work of preservation and enhancement of the historic town.[23] It is in this tradition that we would like to place our proposal for a temporary arrangement, with its limitations.

23 We think, in this particular case, at what has been proposed in G. Samonà, *Caratteri morfologici del sistema architettonico di Piazza San Marco*, in *Piazza San Marco. L'architettura, la storia, le funzioni*, G. Samonà ed., Padova 1970, p. 38.

Part IV
Mathematics and Literature

Models of Passion

Laurent Derobert

Introduction

Existential mathematics is an algebraic model of life wanderings. Etymologically, algebra means to reduce a fracture.[1] And fractures reduction between worlds and beings, between dreams and realities, is the object of existential mathematics. The following pages explore models of passion in this framework: the pursuit of love expressed in a mathematical language.

Axioms

Let l be a subject and L its universe.

l is the vector of three beings that define its identity: real $\bar{l}$, experienced $\tilde{l}$, and ideal $\widehat{l}$. That is, l as it is objectively, l as it perceives itself and l as it would like to be.

L is the matrix of three external worlds of importance to the subject, as above: real $\overline{L}$, experienced $\tilde{L}$ and ideal $\widehat{L}$.

Let us define *maze* as the weighted sum of the distances between the beings and the worlds of a subject. The maze will be represented by the symbol Λ and expressed through the following formula:

$$\underset{Maze}{\Lambda = k_1\left[\alpha_1\left(\tilde{l},\widehat{l}\right)+\beta_1\left(\widehat{l},\bar{l}\right)+\gamma_1\left(\bar{l},\tilde{l}\right)\right]+k_2\left[\alpha_2\left(\tilde{L},\widehat{L}\right)+\beta_2\left(\widehat{L},\overline{L}\right)+\gamma_2\left(\overline{L},\tilde{L}\right)\right]}$$

The maze indicates the subject's torment, which has to be reduced to a minimum: *Min* Λ.

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[1] The word comes from the Arabic expression "al-jabr" which means the restoration of what has been broken.

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The distances between beings and between worlds, of which the subject is the only measure, are indicated by putting between brackets—thus $\left(\tilde{l}, \widehat{l}\right)$ is the distance between the experienced and the ideal being.

Their derivatives with respect to time give their speeds of transformation: thus $\frac{d\bar{L}}{dt}$ is the speed of transformation of the real world.

The set of coefficients denotes the subject's system of values, in particular k_1 the degree of egoism and k_2 the degree of altruism (here $k_1, k_2 > 0$ and $k_1 + k_2 = 1$).

Let us define *vestal* as the weighted sum of the speed of transformation of the beings and worlds of a subject. The vestal will be represented by the symbol V and expressed through the following formula:

$$\mathrm{V} = |k_1| \left[\overline{\psi}_1 \frac{d\bar{l}}{dt} + \tilde{\psi}_1 \frac{d\tilde{l}}{dt} + \widehat{\psi}_1 \frac{d\widehat{l}}{dt}\right] + |k_2| \left[\overline{\psi}_2 \frac{d\bar{L}}{dt} + \tilde{\psi}_2 \frac{d\tilde{L}}{dt} + \widehat{\psi}_2 \frac{d\widehat{L}}{dt}\right]$$

Vestal

The vestal is the index of mobility which has to be optimized.

Mazes and Vestals of Passion

An absolute passion is translated in the model by a complete reduction of the world to the beloved being l^*. In formal terms, L is replaced by l^*. And yet more than this, to love somebody is to help them give life to the world in which they believe. The entire maze of the beloved one is therefore integrated in the maze of the lover. Idem for their vestals. Thus :

$$\Lambda = k_1 \lambda(l) + k_2 \Lambda^*$$

Passionate maze

$$\mathrm{V} = k_1 \upsilon(l) + k_2 \mathrm{V}^*$$

Romantic vestal

with $\lambda(l)$ the ego-maze of l, Λ^* the entire maze of l^*

and $\upsilon(l)$ the ego-vestal of l, V^* the entire vestal of l^*

Interlocking mazes of lovers

If two subjects are impassioned with each other, the following system of interdependence results.

For l:

k_1: degree of egoism *a priori*
k_2: degree of altruism *a priori*, for l^* and L^*

K_1: degree of egoism *a posteriori*
K_2: degree of altruism *a posteriori*, for l^*

And reciprocally for l^*:

k_1^*: degree of egoism *a priori*
k_2^*: degree of altruism *a priori*, for l and L
K_1^*: degree of egoism *a posteriori*
K_2^*: degree of altruism *a posteriori*, for l only

$$\left.\begin{array}{l}\Lambda \;\; = k_1\lambda(l) + k_2\Lambda^* \\ \Lambda^* = k_1^*\lambda\,(l^*) + k_2^*\Lambda\end{array}\right\} \iff \begin{cases} \Lambda = \frac{k_1}{1-k_2k_2^*}\lambda(l) + \frac{k_2k_1^*}{1-k_2k_2^*}\lambda\,(l^*) \\ \qquad = K_1\lambda(l) + K_2\lambda\,(l^*) \\ \Lambda^* = \frac{k_1^*}{1-k_2k_2^*}\lambda\,(l^*) + \frac{k_1k_2^*}{1-k_2k_2^*}\lambda(l) \\ \qquad = K_1^*\lambda\,(l^*) + K_2^*\lambda(l) \end{cases}$$

Confusion of passionate mazes[2]

Thus the interlocking of passionate mazes increases the consideration each lover accords to themselves. Your love (k_2^*) increases the interest I accord to myself ($k_2^* > 0 \Rightarrow K_1 > k_1$). Reciprocally, the more I love you (k_2), the more you cherish yourself (K_1^*).

Among other corollaries of these mixes, we have:

$$\frac{\partial k_2}{\partial K_2^*} < 0$$

The more I love you
the less you cherish me

$$\frac{\partial K_2}{\partial k_2^*} < 0$$

The less I cherish you
the more you love me

Remark 1 These results are valid in the case where the two lovers are positively passionate about each other. The results change when one of the two has a coefficient of negative altruism (cf. jealous love).

Remark 2 These results are pertinent only in the closed circuit of a passionate relationship: one's world is fundamentally the other person. Here again the results change if the passion of one of the lovers opens to other people or objects (cf. plural loves).

[2] Demonstration: $\Lambda = k_1\lambda(l) + k_2\Lambda^*$ and $\Lambda^* = k_1^*\lambda(l^*) + k_2^*\Lambda$ Hence $\Lambda = k_1\lambda(l) + k_2[k_1^*\lambda(l^*) + k_2^*\Lambda]$ i.e. $\Lambda(1 - k_2k_2^*) = k_1\lambda(l) + k_2k_1^*\lambda(l^*)$ QED if $k_1k_2 \neq 1$

Remark 3 Finally, the affective coefficients here are not absolute but relative to the quantum of attention and affection that is given, of which the lovers are the vectors. The results change if this quantum χ changes over time (cf. Modulations Acqua Alta).[3]

Correlation of Romantic Vestals

Equally for the vestals, the romanticism unites the vitality of the lovers.

$$\left.\begin{array}{l} \mathrm{V} \;\; = k_1 \upsilon(l) + k_2 \mathrm{V}^* \\ \mathrm{V}^* = k_1^* \upsilon\,(l^*) + k_2^* \mathrm{V} \end{array}\right\} \iff \left\{\begin{array}{r} \mathrm{V} = \frac{k_1}{1-k_2 k_2^*} \upsilon(l) + \frac{k_2 k_1^*}{1-k_2 k_2^*} \upsilon\,(l^*) \\ = K_1 \upsilon(l) + K_2 \upsilon\,(l^*) \\ \mathrm{V}^* = \frac{k_1^*}{1-k_2 k_2^*} \upsilon\,(l^*) + \frac{k_1 k_2^*}{1-k_2 k_2^*} \upsilon(l) \\ = K_1^* \upsilon\,(l^*) + K_2^* \upsilon(l) \end{array}\right.$$

Mixture of passionate vestals

The interdependence of vestals amplifies the vital energy that each lover invests in themselves. Thus, as with a couple dancing, each partner will give increased consideration to the vigour of their steps in order to better lead the other. Among other corollaries of the mixture of vestals, we have:

$$\frac{\partial k_2^*}{\partial K_1} > 0$$

The more you love me
the more I allow myself to dance

$$\frac{\partial k_2}{\partial K_2^*} < 0$$

The less I love you
the more I allow you to dance

$$\frac{\partial k_2}{\partial K_2} > 0$$

The more I love you
the more you allow me to dance

Passionate Twists and Turns

The above blueprint is valid for an absolute and ethereal love. Yet passion provokes tensions of obsessive desire to transform the other: the fantasies each lover projects onto the other must thus be included in our understanding of the maze.

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[3] Which will be the object of a specific study in Venice in January 2015.

Let f_{l^*} and f_l^* be the following fantasized beings:

$$f_l^* \begin{cases} \tilde{f}_l^* : l^* \text{ seen by } l \\ \widehat{f}_l^* : l^* \text{ dreamed by } l \end{cases} \qquad f_{l^*} \begin{cases} \tilde{f}_{l^*} : l \text{ seen by } l^* \\ \widehat{f}_{l^*} : l \text{ dreamed by } l^* \end{cases}$$

$$\Lambda = \begin{cases} k_1 \left[\alpha_1 \left(\tilde{l}, \widehat{l}\right) + \beta_1 \left(\tilde{l}, \bar{l}\right) + \gamma_1 \left(\bar{l}, \tilde{l}\right)\right] \\ + k_2 \left[\alpha_2 \left(\tilde{l}^*, \widehat{l}^*\right) + \beta_2 \left(\widehat{l}^*, \bar{l}^*\right) + \gamma_2 \left(\bar{l}^*, \tilde{l}^*\right)\right] \\ + k_3 \left[\eta_1 \left(\widehat{f}_l^*, \bar{l}^*\right) + \eta_2 \left(\tilde{f}_l^*, \bar{l}^*\right) + \eta_3 \left(\widehat{f}_l^*, \widehat{l}^*\right) + \eta_4 \left(\widehat{f}_l^*, \tilde{l}^*\right) \right. \\ \quad \left. + \eta_5 \left(\widehat{f}_l^*, \tilde{l}^*\right) + \eta_6 \left(\widehat{f}_l^*, \widehat{l}^*\right) + \eta_7 \left(\widehat{f}_l^*, \widehat{l}_l^*\right)\right] \\ + k_4 \left[\mu_1 \left(\widehat{f}_{l^*}, \bar{l}\right) + \mu_2 \left(\tilde{f}_{l^*}, \bar{l}\right) + \mu_3 \left(\widehat{f}_{l^*}, \widehat{l}\right) + \mu_4 \left(\tilde{f}_{l^*}, \tilde{l}\right) + \mu_5 \left(\widehat{f}_{l^*}, \tilde{l}\right) \right. \\ \quad \left. + \mu_6 \left(f_{l^*}, \widehat{l}\right) + \mu_7 \left(\tilde{f}_{l^*}, \widehat{f}_{l^*}\right)\right] \end{cases}$$

Passionate maze informed by the lovers' fantasies[4]

k_1: degree of pure egoism
k_2: degree of pure altruism
k_3: degree of ego-altruism (oneself through another)
k_4: degree of alter-egoism (another through oneself)

Lovers try to bring together their beings and fantasies by acting as both Pygmalion and Galatea, both sculptor and sculpture. Each of the lines of the preceding formula will be known as 'twists and turns'. For example:

$$\lambda\left(f_l^*\right) = \left[\eta_1 \left(\widehat{f}_l^*, \bar{l}^*\right) + \eta_2 \left(\tilde{f}_l^*, \bar{l}^*\right) + \eta_3 \left(\widehat{f}_l^*, \widehat{l}^*\right) + \eta_4 \left(\tilde{f}_l^*, \tilde{l}^*\right) + \eta_5 \left(\widehat{f}_l^*, \tilde{l}^*\right) \right.$$
$$\left. + \eta_6 \left(\tilde{f}_l^*, \widehat{l}^*\right) + \eta_7 \left(\tilde{f}_l^*, \widehat{f}_l^*\right)\right]$$

Twist and turn of the lover's fantasy

$$\lambda\left(f_{l^*}\right) = \left[\mu_1 \left(\widehat{f}_{l^*}, \bar{l}\right) + \mu_2 \left(\tilde{f}_{l^*}, \bar{l}\right) + \mu_3 \left(\widehat{f}_{l^*}, \widehat{l}\right) + \mu_4 \left(\tilde{f}_{l^*}, \tilde{l}\right) + \mu_5 \left(\widehat{f}_{l^*}, \tilde{l}\right) \right.$$
$$\left. + \mu_6 \left(\tilde{f}_{l^*}, \widehat{l}\right) + \mu_7 \left(\tilde{f}_{l^*}, \tilde{f}_{l^*}\right)\right]$$

Twist and turn of the beloved's fantasy

At this point it is necessary to refine the definitions of the coefficients of egoism and altruism. Let the synthetic formula of the phantasmagoric maze be:

[4] The above formula can be extended to n loves with the subsequent complications that we can imagine.

$$\Lambda = k_1\lambda(l) + k_2\lambda\left(l^*\right) + k_3\lambda\left(f_l^*\right) + k_4\lambda\left(f_{l^*}\right)$$
Abridged form of the passionate maze

k_n is the part of attention and affection a subject accords to themselves whether through their beings (k_1) or through their worlds (k_3). For Narcissus the world is simply a mirror.

$$k_n = k_1 + k_3$$
Narcissus′ degree

Inversely k_e is the part of attention and affection directed at the world whether through its matrix (k_2) or through the self (k_4). The voice of Echo is simply a reflection.

$$k_e = k_2 + k_4$$
Echo′s degree

By definition the maze is the sum of its twists and turns. We could also characterize it succinctly according to its k values. Thus the general form of narcissistic altruism:

$$k_3 \simeq 1 \quad \Lambda \simeq \lambda\left(f_l^*\right)$$
Maze of a narco – altruist or Pygmalion subject

or that of altruistic egoism:

$$k_4 \simeq 1 \quad \Lambda \simeq \lambda\left(f_{l^*}\right)$$
Maze of a echo – egoist or Galatea subject

Pygmalion and Galatea mazes

The woman I loved - I wanted her to fulfil my vision of the world and to share my ideal of an Italian princess. Of course I wanted her to be happy, but I passionately wished her to fit these insane archetypes, ignoring the true identity she scarcely dared to reveal. The ending is obvious. Nonetheless I'll translate it into a mathematical law: when fantasy becomes obsessively inflexible the Galatea and Pygmalion mazes are destined for catastrophe.

The probability that a Pygmalion with inflexible fantasies will desolate his lover is close to 1.

$$\left.\begin{array}{l}\forall\,(t_1, t_2) \in T, t_1 < t_2 \\ T : \text{time of passion}\end{array}\right] \left.\begin{array}{c} k_3 \simeq 1 \\ f_{l_{t_1}}^* = f_{l_{t_2}}^* \end{array}\right\} \Rightarrow P\left(\Lambda_{t_2}^* > \Lambda_{t_1}^*\right) \simeq 1$$
If you try too hard to sculpt another you risk losing them

The Galatean reciprocal is equally verifiable.

$$\left.\begin{array}{r}\forall\,(t_1,t_2)\in T, t_1<t_2\\ T:\text{time of passion}\end{array}\right]\left.\begin{array}{r}k_4\simeq 1\\ fl^*_{t_1}=fl^*_{t_2}\end{array}\right\}\Rightarrow P\left(\Lambda_{t_2}>\Lambda_{t_1}\right)\simeq 1$$

If you allow another to sculpt you too much you risk losing yourself

Index of Corto and Ariadne's Variable

To be loved by those we love divides the maze; inversely, unhappy love multiplies it. Ariadne's variable, A, is infused with the sentiment of loving and being loved. And Corto's index, Ω, is the supreme index of anguish represented by the following formula:

$$\Omega=\frac{\Lambda}{\mathrm{A}}$$

Corto's equation

Corto's Index

The Corto increases with the maze and diminishes with the sentiment of shared love.

$$\Omega=\frac{\Lambda}{\mathrm{A}}\quad\frac{\partial\Omega}{\partial\mathrm{A}}<0$$

Anguish is inversely proportional
to the sentiment of being loved by those we love

The ultimate point is that well-being is attained through the resolution of the Corto. To do this either we reduce the numerator or increase the denominator, minimizing the maze and/or elevating the Ariadne (relieving anxiety and learning to love).

$$\min\left[\Omega=\frac{\Lambda}{\mathrm{A}}\right]$$

Minimization of the Corto

For those who are no longer able to reduce their mazes the only solution is to multiply love in order to ease their torments. But those who fear emotional instability frequently prefer to work on the reduction of their mazes. There are two ways to minimize anxiety: reducing its multipliers and augmenting its dividers.

Ariadne's Variable

Corto's sensitivity to the notion of requited love, embodied by Ariadne's variable, is difficult to model. For this sentiment is highly versatile: its impact differs according to each person and moment. We can nonetheless specify a set of properties that characterize Ariadne's variable.

Let k_2 be the degree of affection for the beloved and let k_2^* be its reciprocal.

Property of Montague also called "Nature Boy Property". *The greatest thing, you'll ever learn, is just to love and be loved in return.* The highest value of A is obtained when the love is absolute and absolutely shared.

$$\mathrm{Sup}\left[\mathrm{A}_{\left(k_2,k_2^*\right)}\right] = \mathrm{A}_{(1,1)}$$

Ariadne′s variable culminates in the reciprocity of consummate love

Property of Capulet The weakest value of A is obtained from the feeling of utter abandonment.

$$\mathrm{Inf}\left[\mathrm{A}_{\left(k_2,k_2^*\right)}\right] = \mathrm{A}_{(1,0)}$$

The indifference of those we love minimizes Ariadne′s variable

Constraint of Theseus To be loved by someone we don't love or to abandon somebody that loves us increases the torment of the maze.

$$k_2^* \simeq 1 \Rightarrow \lim_{k_2 \to 0} \mathrm{A}_{\left(k_2,k_2^*\right)} < 1 \left(\Rightarrow \Omega = \tfrac{\Lambda}{\mathrm{A}} > \Lambda\right)$$

Eminent love increases the anxiety of those who don′t requite it

This constraint of Theseus in conjunction with the property of Capulet forms a family of functions of Ariadne known as the Naxos ($\aleph$), in which relationships that are significantly asymmetrical are doomed.

$$\mathrm{A} \in \aleph \Rightarrow \exists \tau < 1 : \left|k_2 - k_2^*\right| > \tau \Rightarrow \mathrm{A}_{\left(k_2,k_2^*\right)} < 1$$

Family of Naxos

hyper asymmetrical loves are condemned to anxiety

We can identify another family of functions called Ithaca ($\Im$) by specifying the limits of Ariadne in addition to the properties of Capulet and Montague.

To get a taste of these limits think of Penelope and Ulysses, the sighs that time could not erase and the vows unbroken despite the promise of immortality.

$$\mathrm{A} \in\Rightarrow \begin{cases} \lim\limits_{k_2^*\to 1} \mathrm{A}_{\left(1,k_2^*\right)} = \infty \\ \lim\limits_{k_2^*\to 0} \mathrm{A}_{\left(1,k_2^*\right)} = 0 \end{cases}$$

Family of Ithaca
the limit of Ariadne is infinite or infinitesimal
as the passion is symmetrical or asymmetrical

Illustration Imagine a lover whose function of Ariadne belongs to the family of Ithaca. Madly in love, he receives a letter assuring him of the reciprocity of his sentiments:

$$k_2 \approx 1; k_2^* \approx 1 \Rightarrow \mathrm{A}_{\left(k_2,k_2^*\right)} \approx \infty$$

$$\Rightarrow \Omega = \frac{\Lambda}{\mathrm{A}} \approx 0$$

Then on checking the envelope he discovers that the letter is destined for another:

$$k_2 \approx 1; k_2^* \approx 0 \Rightarrow \mathrm{A}_{\left(k_2,k_2^*\right)} \approx 0$$

$$\Rightarrow \Omega = \frac{\Lambda}{\mathrm{A}} \approx \infty$$

Even the weightiest torments of the maze seem like a feather to the requited lover. *A contrario*, the sentiment of abandonment swells even the smallest of mazes.

The model of existential mathematics, reduced to its simplest form, can be expressed thus:

$$\min\left[\Omega = \tfrac{\Lambda}{\mathrm{A}}\right]$$

Minimization of the Corto
through Ariadne expansion
and Maze reduction

The Wolf and the Street: Narrative Encounters with Mathematics

Matilde Marcolli

Introduction

I was invited to this conference to present the results of an old experiment, a twenty year old exploration of narrative and mathematics, in the form of a collection of fairy tales.

I began writing the stories of *Racconti per il lupo* (Tales for the Wolf) in 1986, when I was a 16 year old student of the *Liceo Classico*, more occupied with ancient Greek texts and philosophy than with mathematics. I finished the collection when I was about to graduate in Theoretical Physics, at the University of Milano, in 1993.

Those few years spanned enormous transformations, at the personal level, going through the passage from late adolescence to adulthood, and from being a student of classical languages to a professional physicist, as well as on the larger scale of society and the world: those were the years that marked "the end of the short century", with all the upheaval, excitement and anxiety that came with it. In their minuscule cameo scale, the mathematical stories I was putting together, act as a small fragmented mirror of larger events.

The stories, written in Italian, and illustrated by a series of collages I prepared in the style of Max Ernst, are now available online at the publisher Lulu.com, along with some of my more recent writings.

In this presentation, I will try to describe the main ideas behind that old attempt at conveying in a narrative form some mathematical concepts, and I will contrast the spirit of that early encounter with mathematics, with the very different spirit in which I came to see the mathematical profession nowadays, after twenty years of experience. The latter is best represented in some of my more recent writings, especially the lyric prose collection *Street Science*, composed in 2013, illustrated by mathematical street art graffiti, also available online from the same publisher.

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Tales for the Wolf

I will describe briefly the structure of the stories in *Racconti per il Lupo* and some of the main narrative themes. The work consists of 18 short stories, collected under the title *Matematica e altre fantasie* (Mathematics and other fantasies).

The titles of the individual stories are: *The breath of the Aegean*, *Two-voice invention*, *Desert rose*, *Klein's Easter*, *Transcendental love*, *Little Red Hood*, *Sergels torg*, *Venetian shadows*, *The projective oval office*, *The Vodnik in the Ljubljanica*, *Stockholm fish*, *Ghost*, *The cold night of Helge von Koch*, *The ice cream seller*, *The hyperbolic fashion designer*, *The crazy puzzle of the Republics*, *The monkey of randomness*, *Möbius band*.

Mathematical Themes

Each story is followed by a couple of pages of discussion, where the main mathematical idea contained in the story if briefly recalled and explained. These are, respectively, Gödel's incompleteness theorem, underlying the story "The breath of the Aegean", the Banach–Tarski paradox in "Two-voice invention", Penrose's aperiodic tilings in "Desert rose", non-orientable surfaces in "Klein's Easter" and "Möbius band", Cauchy sequences and the construction of real numbers in "Transcendental love", Banach–Caccioppoli's fixed point theorem in "Little Red Hood", Banach and Hilbert spaces in "Sergels torg" and "Stockholm fish", fractal dimensions in "Venetian shadows" and "The cold night of Helge von Koch", the classification of conics in "The projective oval office", the optimal stopping problem of probability theory in "The Vodnik in the Ljubljanica", the group of Lorentz transformations in "Ghost", the geometry of quadrics in "The ice cream seller", hyperbolic geometry in "The hyperbolic fashion designer", the chromatic number and coloring problems in "The crazy puzzle of the Republics", random sequences and Kolmogorov complexity in "The monkey of randomness".

As one can easily see, the range of topics includes more elementary mathematics, such as the classification of two-dimensional surfaces, topics from students standard curriculum, like Dedekind's construction of real numbers, or the notions of Hilbert and Banach norms, of conics and quadrics. There are also topics that are more frequently seen in mathematical popularization, like fractals. There is hardly any mention of more sophisticated mathematics, except for Gödel's theorem, another favorite of mathematical popularization. The choice of topics was not planned. I just wrote about any mathematical topic I came across as a student in those early days that captured my curiosity. I turned them into stories, and used the stories as a way of communicating the mathematical ideas to a larger public (back then consisting mostly of a circle of close friends).

The stories are accompanied by illustrations, which I made as collages, in the style of Max Ernst. They are simpler illustrations than the more elaborate collages that constitute the second part of the book, the visual narrative "Wolf in the Fold", about which I'll say a few words below, and they are meant only to capture the

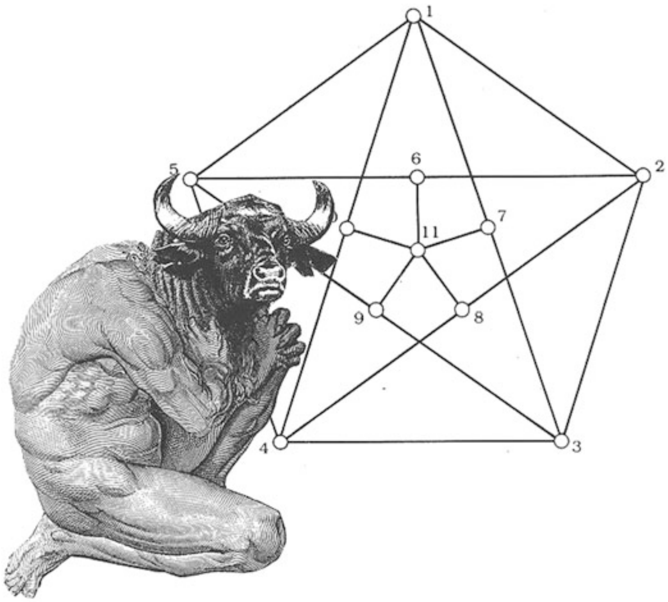

Fig. 1 Illustration from "The breath of the Aegean"

spirit of the story, not to provide a story in themselves (Fig. 1). The reason for this choice of style is explained briefly in the introduction, where I wrote: "The images are a hommage to Max Ernst, for no other reason than in memory of that *Surrealism in the service of Revolution*, or perhaps only because Surrealism and Mathematics always make for a happy combination".

Little Red Hood and the Quest for Identity

Besides the mathematical themes mentioned above, the stories are full of references to the geopolitical events of the time (late 1980s and early 1990s) when they were written.

The story that gives the title to the collection is a retelling of the fable of Little Red Hood, with some twists in the story, which ends with the little girl and the mathematician wolf eloping and living happily ever after. The "Tales for the Wolf" are, supposedly, the stories with which Little Red Hood will entertain her lover wolf, during their happy days in the depths of the forest. The same story contains a parallel political theme, on the effective dissolution of the traditional Italian political Left in 1991 and the vacuum that it left behind, and the difficult quest for identity of its survivors.

The latest days of the Cold War and its aftermaths also make an appearance in the stories "The projective oval office" and "The crazy puzzle of the Republics".

Theseus, Oedipus, Pindar and the Ancients

Another theme crisscrossing through the stories of *Racconti per il Lupo* is Greek mythology and references to ancient Greek culture. This is inevitable, since I underwent the typical Italian intellectual upbringing, with a good scholarly education in the Classics, Latin and ancient Greek. I had a serious interest in the Greek language and culture, to the point that I seriously entertained the idea of Philology of ancient languages as an alternative career choice to Physics and Mathematics. (Well, there's still time.)

So the story about Gödel's incompleteness theorem is set inside the framework of the Greek myth of Theseus, with a few hidden references to writings of Károly Kerényi and Bruno Snell. Similarly, "The monkey of randomness" has an ancient Greek setting, including Greek citations from Sophocles "Oedipus Rex" and Pindar's first Pythian.

Cities of the Eastern Nights

Another kind of mythology permeates the stories, a mythology of cities: Stockholm, Budapest, Ljubljana. Northern and Eastern European cities, Mitteleuropa, as I had experienced it, in a flurry of train traveling outside of our national boundaries, in that brief period of time in between when frontiers became suddenly permeable and when wars began to ravage the Balkans. A brief, almost magical, encounter which is embodied in a dance of cities and water: fountains in Stockholm, the Danube passing between Buda and Pest, and rivers in Prague and Ljubljana inhabited by magical Vodniks.

Flowers from Baghdad

The story "Desert rose" was written in 1991, in the wake of the starting of the first Gulf War. It contrasts the Baghdad of the historical Islamic Renaissance, the center of culture, science, and art, with the modern war ravaged, bombed city. Sadly, twenty years later, this is the only story that has not lost its relevance in terms of immediate reference to contemporary political events (Fig. 2).

The Möbius-band-shaped poem that forms the last story is another hommage to a middle eastern setting, a touch of Orientalism perhaps, to which our country of rich Arabic heritage is easily susceptible (just think of Pasolini's majestic filmic rendering of "The Arabian Nights"). In fact, the form of that last poem is more directly inspired by a song by the Italian singer songwriter Francesco Guccini, set in a similar desert setting, which in turn elaborates on the biblical verse, "Watchman, how much is left of the night?".

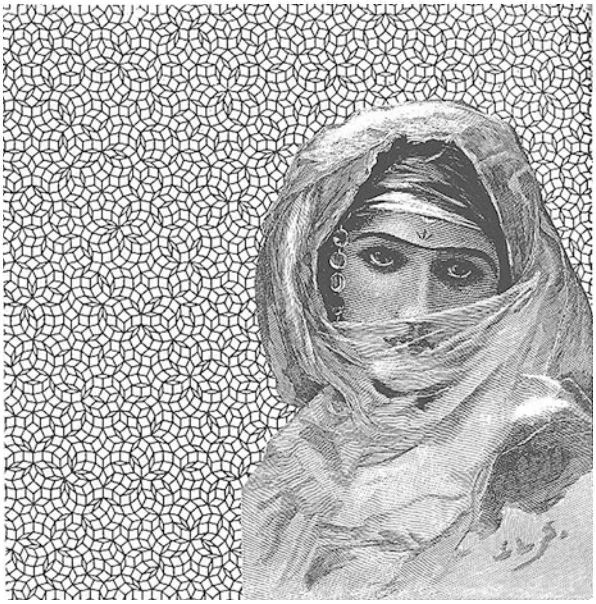

Fig. 2 Illustration from “Desert rose”

Wolf in the Fold

The second half of the book *Racconti per il Lupo* consists of a visual collage, titled *Wolf in the Fold*, conceived as a hommage to Max Ernst’s famous visual “collage-novel” *Une Semaine de Bonté*. It consists of 19 collages, each accompanied by a short poem. The poems are themselves collages, made of titles of paintings by the main artists of Surrealism.

The “story” narrated in “Wolf in the Fold”, in as much as it makes sense to speak of storyline in Surrealist collage-novels, is the rest of the story of Little Red Hood and the Wolf, after their happy first encounter narrated in one of the short stories in the first half of the book. It is a story of love and mathematics, of happy moments of collaboration and moments of deep suffering and despair. All in all, it is an open-ended cautionary tale about the delicate balance between infatuation, especially in relation to those people we perceive as role models in the early days of our mathematical apprenticeship (and sometime in the later stages of our life as well) and the potential disillusionment that always lurks in the act of projection, in the Jungian sense, of our aspirations onto others.

It is, in a way, nothing more than the standard story of the inevitable voyage from love to disillusionment, set in the framework of a mathematician’s life. While the stories of the first half of the book are about mathematical concepts, embodied in a framework of narrative, the visual collage *Wolf in the Fold* is already something else, more akin to what my later writings will become, not a story of mathematics but

Fig. 3 From the surrealist collage-novel "Wolf in the Fold"

one of mathematicians, not a cheerful story of ideas, but a darker story of individuals (Fig. 3).

It is a tragic mistake, which is easy to make over and over again during a lifetime of scientific enquiry, to measure one's value, interests, aspirations and achievements upon a meter set by somebody else, someone we admire, perhaps, and would like to invest of the power of life and death over a path which is ultimately solely our own to plot and govern. While it may seem at first rewarding to receive confirmation and encouragement from a trusted source, ultimately in science nothing good can possibly come from any form of authority principle. Science is anarchist by its very nature.

In retrospect, the disillusionment repeatedly caused by a sequence of failed attempt at searching for a "magical other", with whom to share the intellectual adventure of exploration of the boundaries of the mathematical unknown, are perhaps one of the main factors in the progressive darkening of my perception of the profession. Far from being the collective, shared, progressive enterprise I had imagined it to be at around the time when I was writing the stories of *Racconti per il Lupo*, mathematics as a profession revealed itself to be largely a callous and vicious environment, where enthusiasm is systematically snuffed out, where trust in other human beings is nearly always misplaced, and no real sense of common purpose

exists. There is no “other”, there is no “sharing”. As I wrote elsewhere (in my theater play *The Somber Science*): “It would be so easy to create an illusion of meaning in our lives: it’s enough to have an ideal to fight for, a desire to change the world, a community to belong to. It is so tempting to get validation, support and a sense of purpose, through these outlets: Party, Commune, Collective. Science is a harsher meter to confront oneself with: there is no easy way to reassure oneself, no red banners to wave in the wind. We are all alone in front of the terrifying silence of the universe. Those rare moments when one can share the experience of understanding with another fellow traveler are precious, unique.”

Night Songs

The introduction to *Racconti per il Lupo* consists of two pieces, respectively called *Canzone di Notte, N.1* and *Canzone di Notte, N.2* (Night Songs, N.1 and N.2). The titles are a direct reference to a series of songs by the same name, by the Italian singer songwriter Francesco Guccini. These introductions were later additions, written in 2011, when I came across the old manuscript of *Racconti per il Lupo* and decided to publish it. They attempt to summarize the cultural context in which those stories were written, the leftist Italian *intelligentsia*, which was going through a period of crisis and restructuring in the late 1980s and early 1990s. Pasolini had variously referred to that cultural milieu as *“il paese nel paese”* (the country within the country), a very tightly wound community, encompassing the vast majority of Italian intellectuals (writers, artists, musicians, scientists, philosophers) over several decades of postwar history. That environment, which I grew up in and belonged to, effectively ceased to exist, with the transition of the early 1990s that ushered 20 years of populist governments pandering to the extreme right, and no longer held in check by a strong, well organized and intellectually capable Left. I was no longer living in Italy by then, and I was thankfully spared the experience. My “Night Songs” are songs of remembrance, which are meant to set the larger stage on which the otherwise very light and humorous short stories are taking place.

Street Science

Let me now fast forward ahead 20 years, from 1993, when the last of the stories of *Racconti per il Lupo* was written, to 2013, when I wrote the collection of lyric prose *Street Science*.

The protagonist of *Street Science* is no longer mathematics, but the experience of mathematics as a profession, articulated as a journey in 12 episodes, from student to full professor, through graduate school at the University of Chicago, a postdoctoral experience at MIT, a few years at a Max Planck research institute, finally landing with my present job at Caltech.

Fig. 4 Illustration from "Street Science"

Mathematics itself is present in *Street Science* in the illustrations that accompany the lyrics (Fig. 4). Unlike the illustrations of *Racconti per il Lupo*, these do not try to give a pictorial symbolic image that evokes the atmosphere of a story, and where mathematics is incorporated as a symbol, but they deal directly with mathematical formulae, without context or metaphor. Advanced mathematics written as urban graffiti: the metaphor here is in viewing the mathematical community through the eyes of the asphalt jungle.

Street Art and Street Signs

The title of the collection is clearly meant to bring to mind two associations. The first is with *Street Art*, which is the general term referring to urban graffiti artists, a form of art that is associated to political protest, to the marginalized inner city youth, cut out from mainstream society. The term *Street Science* positions the author as an outsider to the scientific establishment, even though formally a part of it by official curriculum and affiliation. Street Art is primarily a form of protest and revolutionary fight, achieved through the means of art. I similarly perceive my scientific activity nowadays as being a war, fought one small lemma at a time, with each paper the analog of a so called "street art bomb", in the urban jargon, or even just a "tag". The other reference is to the term *Street Signs*, meant as marks of directions, pointers of the way, for those who, in the middle course of life, may have found themselves in a dark forest.

The illustrations that accompany the collection of lyrics are meant precisely as signposts, drawings in the style of street art graffiti, superimposed on a blueprint of the New York subway wagons, representing mathematical formulae taken from various results I worked on, at different times in my mathematical career. They are placed in the text to mark the periods of time the corresponding lyrics refer to with the appropriate mathematical content. So there is gauge theory and noncommutative geometry and motives and quantum field theory, slowly moving towards the singularity of the present moment, where all of these paths seem to have come to abrupt endpoints.

In the Alley of Dark Workshops

Street Science takes off exactly where *Racconti per il Lupo* ends, with a first piece set in 1993, the year when I graduated in Physics and started off on a journey that would cause me to change country, switch from Physics to Mathematics (or at least physically inspired mathematics) and gradually climb up all the ladder of the professional scientific community. The first episode, called *Midnight in the Alley of Dark Workshops*[1] describes exactly the same scene that is portrayed in the two "Night Songs" in the Introduction to *Racconti per il Lupo*.

Rites of Passage

The 12 episodes in which "Street Science" is articulated form a series of rites of passage: *Chicago* is a snapshot of my graduate school experience, and *Mother* is a lyric tribute to my mother, who died of cancer while I was in my last year of graduate school. The next passage is the one that took me from the University of Chicago to MIT, where I did my postdoc: *First stop Wonderland* recalls the early enthusiasm with which I took the new experience, quickly dissipating into more somber feelings. (Wonderland is also the first stop on one of the Boston subway lines.) The next episode, *Gauge theory* follows the growing disillusionment with the gauge theory and low dimensional topology community I nominally belonged to during my MIT years, until the final cut and my move overseas to the Max Planck Institute.

> ... I left unfinished theorems, shadows painted on a wall, a bicycle abandoned in the department hall ...

Paradise regained recalls the difficult restarting from scratch in a new field of research that followed this first (and by no means last) debacle. The lyrics focus on a beautiful friendship that grew during the years I spent at the Max Planck Institute, which did so much to restore my sense of purpose and of identity.

> ... a Gauss measure between art gallery and poetry night, a modular form, a motive, a vision, sometimes a Latin poem. Poetry as a rhythmic articulation of feeling, math as a logical articulation of beauty, a smile to the walkways of memory, lakeshores, and midnights of changing years ...

In the following two pieces, *Leçons de ténèbres* and *California dreaming*, a much darker side of "paradise" manifests itself. *Leçons de ténèbres*, named after the polyphonic lamentations composed for the "tenebrae" services (Gesualdo, Charpentier, Tallis, Couperin), reflects the time when a new and more traumatic turn was forced on me by the unexpected breakdown of a long collaboration I had

[1] To catch the historical reference in this title, it is perhaps useful to mentally translate it into Italian.

invested in enormously for several years. (This event and the deep scars it left was discussed at length in several of my recent writings, including the theater play *The Somber Science* and the collections of essays *Screams of Metal* and *Listening to Golem*.)

California dreaming is a bitter ballad against the German “herrprofessordoktors”, which explains the reasons behind my decision to relocate in California, despite the excellent working environment of the Max Planck institutes and the close friendships and deep human relations I had established there.

The following piece, *Father*, deals with the death of my father, which happened two years after I moved to California, and his uneasy human inheritance, so much echoing other woes described in the other episodes of the *Street Science* collection. The remaining three pieces, *Cipher-Punk*, *Sunset Boulevard*, and *Chaos Ashram* deal with my new life in California, with its promise, with a mixture of marvel and despair, and with all the lose ends of the past still lingering unsolved.

The End?

I have returned time and again to the use of narrative forms as an accompanying commentary on my experiences as a professional mathematician. The original series of short stories *Racconti per il Lupo* is by far the most cheerful and naively enthusiastic of my narrative encounters with mathematics. It was followed by two science fiction novels: *City of Sorrow*, written in 1999 and loosely based on my experiences at MIT, and *Yamaon*, written in 2002 and inspired by the atmosphere at the Max Planck Institute. In 2009, shortly after I moved to California, I wrote the theater play *The Somber Science*, as a way of expressing in written form the traumatic experiences that accompanied my relocation (the same described in *Leçons de ténèbres* and *California dreaming* in the collection *Street Science*). In 2013 I tried to give with *Street Science* a panoramic view of my experiences of the past 20 years, from the vantage point of these last few years of Californian life, as an attempt to come to terms with the profound disillusionment I have by now developed towards the mathematical community. I do not know, at this stage, whether this perspective is likely to shift in coming years; whether the beauty of science itself will continue to prevail over the human ugliness of its practitioners, offering renewed comforting vistas of beauty, capable of rekindling our enthusiasm and motivation after each new pitfall of disillusionment, or whether there is ultimately a breaking point beyond which the homeostasis ceases to restore the system functionality. What I do know is that, either way, the narrative form can be used to represent this process metaphorically and poetically, and it often offers us a unique path towards healing.

Anxious Geometries

Claudio Zanini

Introduction

These short poetic texts, the first ones written—many years ago—as a commentary on representation of imaginary arrangements of volumes and architectures, have as their object geometric entities, and a certain melancholy given to them by their inflection in shadow or disappearance in light. They are infused with aching by the unresolved relations between finite and infinite, between truth, uncertainty and doubt; light and shadow; permanence and the inexorable flow of time.

There are triangles whose sharp angles pierce; parallel lines where one vanishes, leaving the other to wander alone in infinity; plane figures that are victims of deceptive aporias, still others which acquire volumes that are incongruous and disturbing.

The verses are counterpoised by illustrations which are not strictly related to the text, but which nevertheless provide suggestions of the same luminous lightings, the same uncertainties and ambiguities.

They are incomplete figurations of imaginary theorems, perhaps dreamed in Argentine nights by Jorge L. Borges; fragments of faraway geometries, in paradoxical formulations; apocryphal proofs, still unknown and unheard of; obscured by light, barely caressed by shadow, and yet waiting to be plunged into it.

Are these poetic digressions, or reckless metaphors?

In a certain way this cannot be denied; however, given that such suggestions are born of recollections—real, not phantasmal—that emerged from the memory of my studies at university, they have a certain rational foundation.

I remember that I learned, with emotion and surprise, that classical space, the space in whose geometry we and our loved ones quietly walked among everyday objects and in measurable places, began to vacillate and reveal increasingly deep and irreparable crevices, in the first half of the 1,800 s when the new, audacious geometries elaborated by Lobachevsky, Bolyai and Gauss [1] undermined traditional

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Kantian identification between absolute space and Euclidean geometry. The customary, sure 'plane' of reference first revealed unexpected anomalies, then curved, slid, reared up: it can be elliptical or hyperbolical. No geometry can boast of absoluteness, but can only be valid in a certain environment, and must accept the relativity of its object.

Today, in fact, we can move not only along the Euclidean lines of a soccer field, but also—with the right vehicles and instruments, and imagination too—along the dizzy pathways of the geometries of the microcosm and macrocosm, or in the infinite germination of fractals.

Even time, which pulses in our veins, ever fatally present in our lives and those of objects, and from which, however, we remove our thoughts because it 'lacerates and corrodes, and nails and teeth no longer come to our rescue' (see the poem *All'eccessiva vastità*); time too, I said, comes into play as an inescapable dimension, in the space/time continuum that determines the 'field' of reciprocal forces in which the object subsists and becomes, of which the 'field' is a sort of emanation [2].

But time signifies mutation and becoming, and the living organism is, not always knowingly, subject to these motions, which substantiate it and from which it cannot subtract itself.

This had already become clear by around the second half of the 1,800 s, when Ernst Mach [3] distinguished between *Raumbegriff*, conceptual space, and *Seheraum*, space that can be sensed, thus physiological, tactile and immediate. This important distinction was reaffirmed by Ernst Cassirer in his *Philosophie der Symbolischen Formen* (1923–1929) [4]. This is thus the space of the body.

But it was above all the French philosopher Gaston Bachelard [5] who spoke of emotional spaces, warmed by fire and clarified by light, or cold and obscured by shadow. These spaces were not only geometric, but inhabited, defended against enemy forces. They were spaces that were mythical, loved or hated, and thus subjective, where dimensions were uncertain, ambiguous, disorienting, as in certain works by Escher.

This fascinating space/time is alluded to both in poetry analogous to that mentioned earlier, *All'eccessiva vastità*, and in a series of small paintings where, paradoxically, the geometry of the solids, often represented axonometrically, is ephemeral and transparent, while the space around them is heavy, material, enveloping, and in certain points displays the inflammation of lacerations, as if provoked by wounds that cannot be medicated or by indelible combustions (Figs. 1 and 2).

Is this an adventurous incursion into the most fantastic imagery?

Perhaps not, since we are speaking of the space of the body and thus, consequently, of that of anthropology, psychoanalysis, symbolic systems, and so forth.

Just as some poetry contains allusions to the influence that sloping shadows exert on the soul of the geometer in the course of elaborating a mathematical problem, so too in illustrations it can happen that a trace of an imagined theorem emerges from the shadow, or red flares in a pointed vertex. In light of what we have said, it might be less paradoxical than it seems to allude to the fact that one geometric theorem

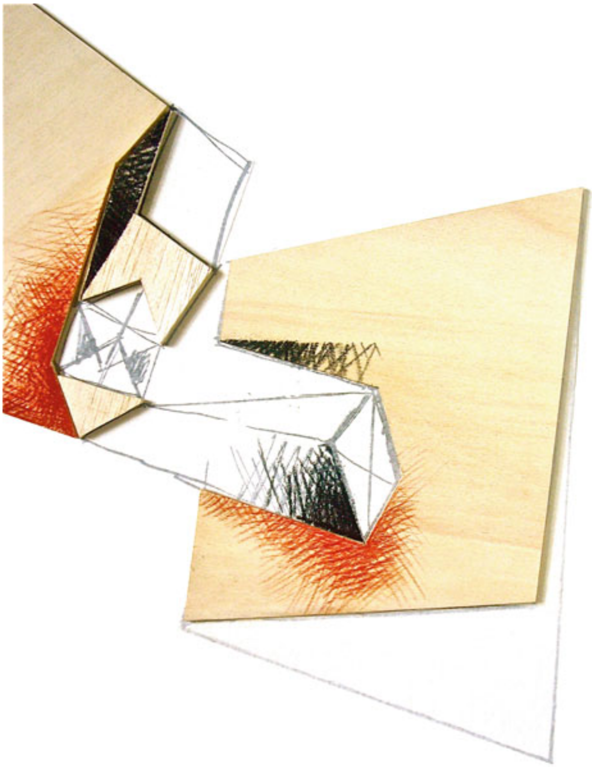


Fig. 1 Incastro di torri difensive

carries the stigmas of a wandering nomadic tribe in the burning hot desert, while another reveals the conceptual thought of a community living in the cold peaks of the Himalaya, and still another, in some way, carries within it the sadness of dusk (Figs. 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12).

(Translated from the Italian by Kim Williams)

On *Quadrato nero* by Kasimir Malevic

So many verses have been inspired by the famous painting *Black Square* of 1914–1915 by the Russian painter Kazimir Malevich, founder of Suprematism, a movement of the Russian avant-garde.

With this work (here liberally interpreted by me), Malevich intended to arrive to the zero degree of painting, that is, the total abandonment of representation, the mimetic relationship with a reality *other than self*. What remains, and what the artist was striving for, are pure perception and absolute sensitivity (Fig. 12).

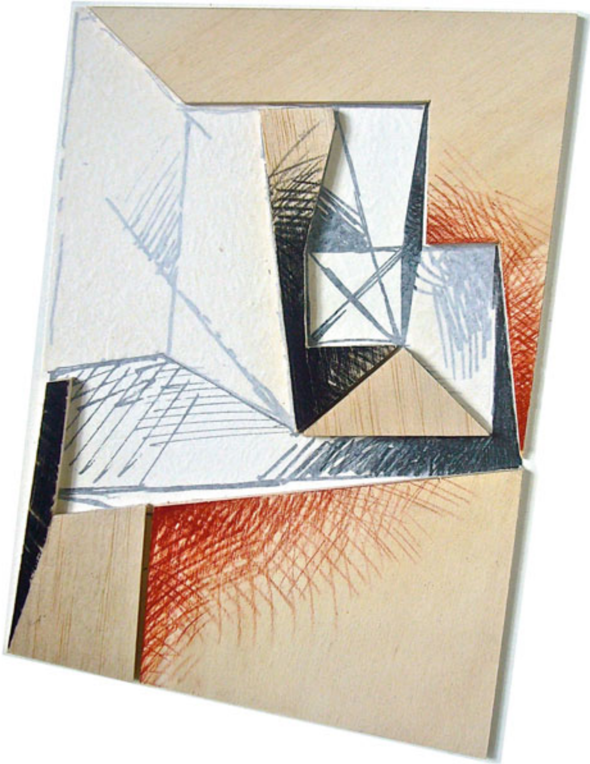


Fig. 2 Cortile racchiuso tra torri difensive

In my pictorial interpretation, the black square is perceived frontally, and is not exactly a square, since there are asymmetries, optical corrections. The vertical sides are slightly longer with respect to the horizontal sides; the space below the square is higher with respect to that above it. There are also other, almost imperceptible deformations. The background is not white but vibrant with material and light, the gray has various tonalities; on top, in the upper middle panel, it deepens to black. Not even the black is absolute, but allows the slightest gleams, fragments of light.

However, in all these interpretations, pictorial as well as poetical, there emerges, from the apparent imperturbability of the black square, margins that evoke something, allude to *other*, and absorbing our gaze, whisper questions to us. In affirming that it signifies nothing and hides no enigma, it raises a subtle anxiety. An archetypal form, the black square evokes, in its dark and almost imperceptible whispering, that most hidden aspect of intimacy that Jung identified as the Shadow. It is the Shadow which is inhabited by removed feelings of guilt, indelible uncertainties of the ego.

(Translated from the Italian by Kim Williams)

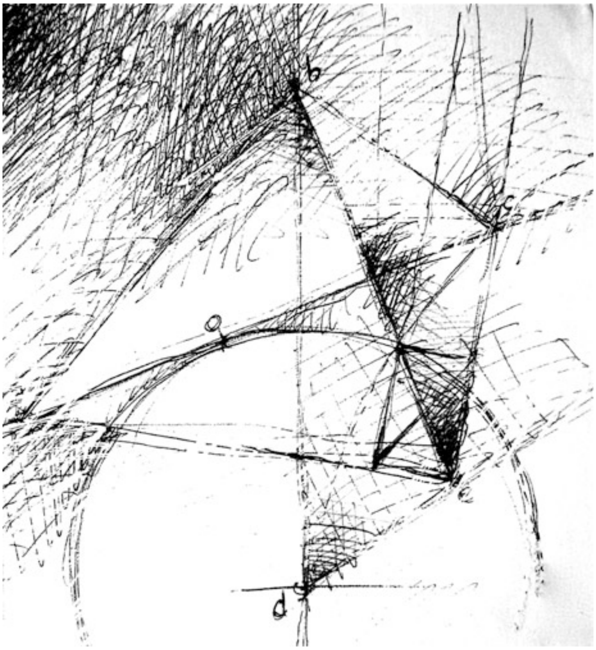


Fig. 3 Frammento di un teorema mai seriamente enunciato

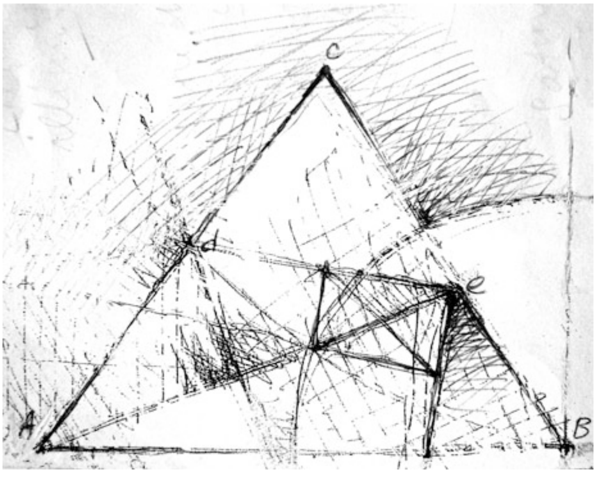


Fig. 4 La base di un triangolo ABC è due volte il raggio del cerchio centrato in B e se dall'ombra che dal fondo . . .

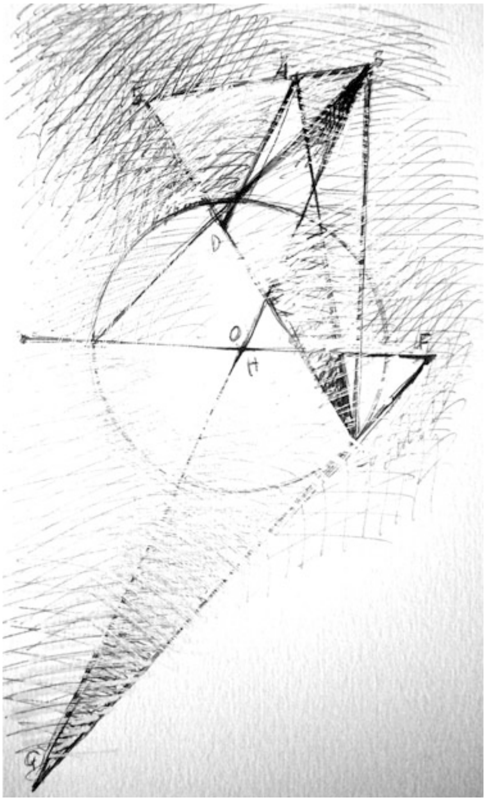


Fig. 5 Se i due triangoli GBC e G'BC s'accendono di luce

Poems/poesie

12

The golden section of the ruler does not voice
the wise cartographer's knowledge,
for shade shuns triangulation
and light isn't but the unquiet dominion
of the winged creatures. Arduous to trace
neat trail in the garden: tracks do clog
in mill-ponds of shade or in fine clarity
of vagueness, of outlandishness.
Among arboreal encumbrances and humid tuffs
entanglements aren't but harsh knots of sorrow.
Let us then yield to a minor outline
as of the skylark's bold map-threading,
to the mocking measure of its singing,

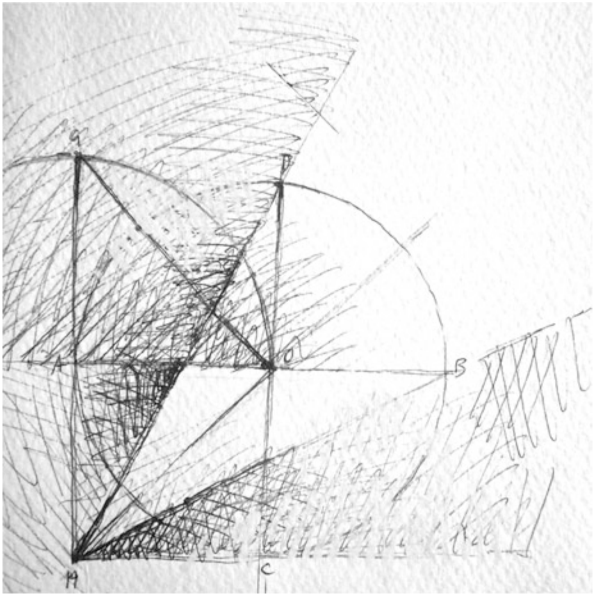


Fig. 6 Frammento di teorema irresoluto sulla sparizione della bisettrice

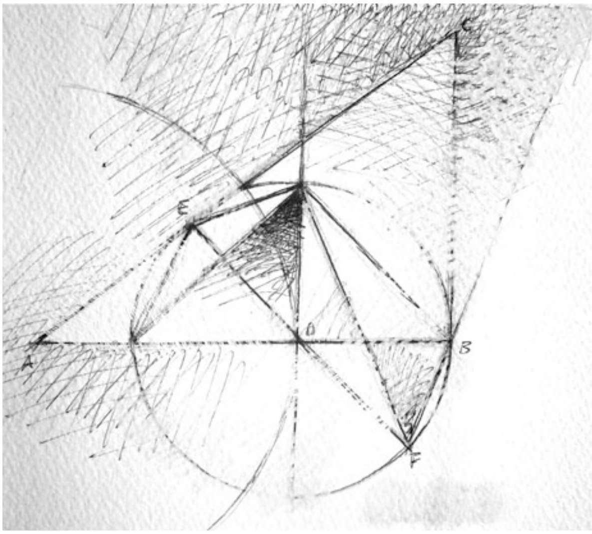


Fig. 7 Teorema di ardua decifrazione, relativo al triangolo rettangolo ABC

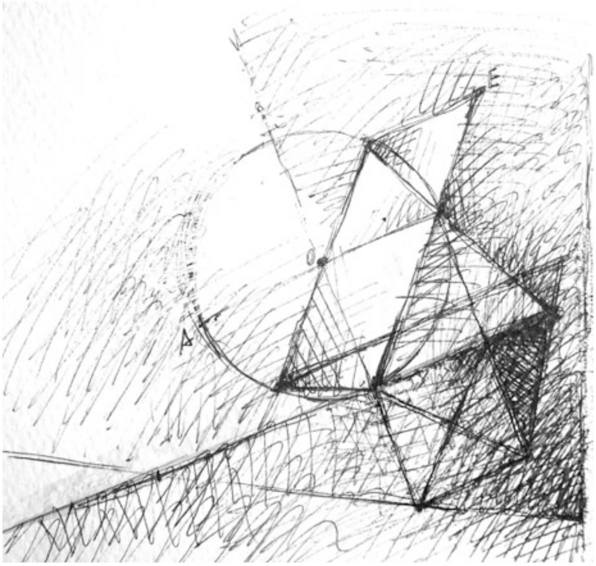


Fig. 8 Teorema apocrifo d'una geometria invasa dall'ombra

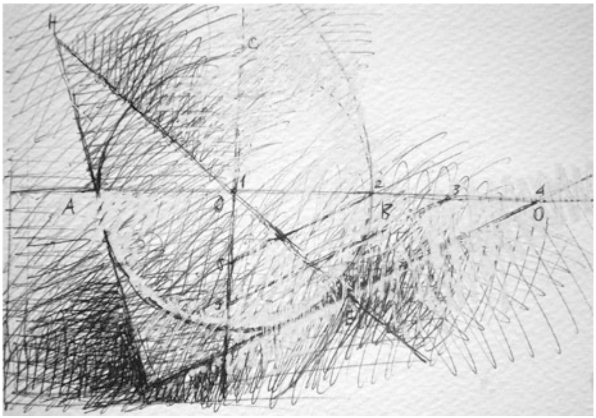


Fig. 9 Luci inattese nella dimostrazione di un teorema poco conosciuto

although almost nothing is left behind
but the uncertain presentiment of a quietude.

12

Non esaurisce l'aurea misura del regolo
nessun sapere del cartografo valente
poiché l'ombra sfugge ad ogni triangolazione
e la luce è dominio irrequieto dei volatili.

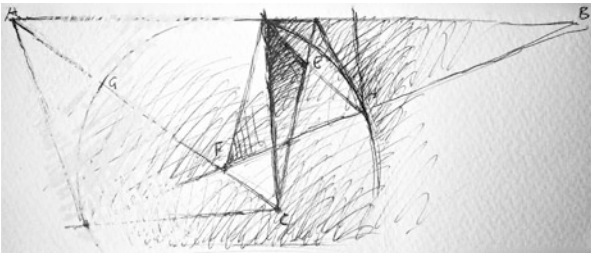


Fig. 10 Frammento d'un teorema sconosciuto e offuscato dalla luce

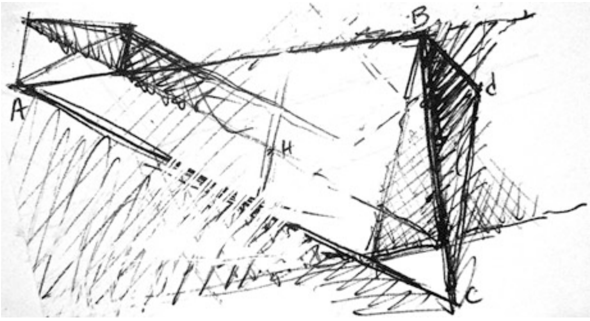


Fig. 11 Teorema enunciato in una lingua ancora sconosciuta

Arduo comporre limpida traccia nel giardino:
s'interrompono i sentieri in gore d'ombra
o in tenue chiarità di lontananze vaghe.
Tra frangenti arborei e zolle umide
sono gli incroci aspri nodi d'afflizione.
Affidiamoci allora al minimo tracciato
della cartografia audace delle allodole:
alla misura irridente del loro canto
sebbene, a noi, quasi nulla resti,
solo il presagio incerto d'una quiete. (Fig. 3)

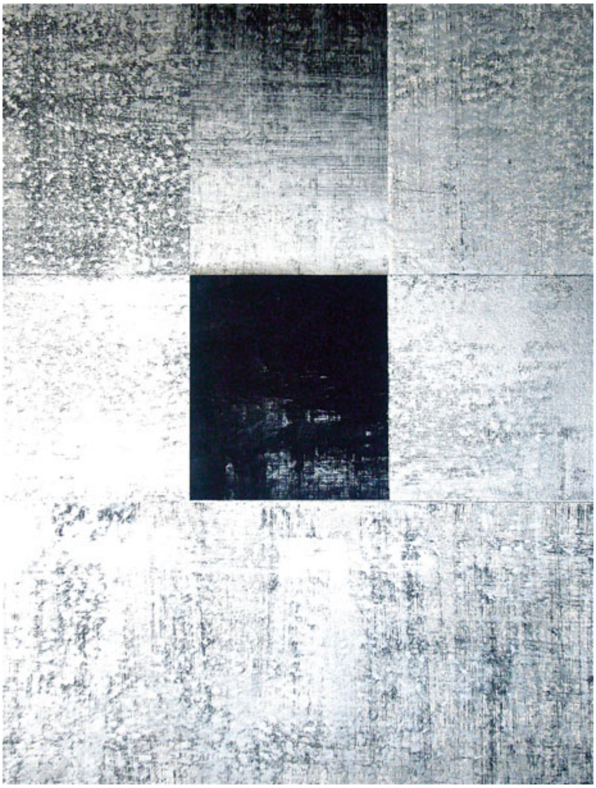
Fragment of a theorem never seriously stated (since it has just emerged from shadow . . .)

Frammento di un teorema mai seriamente enunciato (poiché appena emerso dall'ombra . . .)

17

Metrication is sought by the universe for
pathways to blossom into measures to be trodden,
outlining spaces unto the boundary of void.
Pathways, silver slimes in the time of night,

Fig. 12 Quadrato nero (da Kasimir Malevich)



nights of dreadful eclipses, to go through
in winding exhausted roaming to the extent
of unbecoming spaces, whilst betrayed
by stars once relied upon as fixed.
Uncertain travelers in the frail cavity of memory,
may find it sweet to pause at a hollowed border
overwhelmed by flowing waves of time.

17

Un sistema metrico è necessario all'universo
a dischiudere sentieri in misure percorribili
tracciando spazi certi al limite del vuoto.
Si percorrono sentieri, bave argentee nella notte
in tortuosi andirivieni stremati ai confini
di spazi insostenibili, in notti di tremende eclissi
traditi da quegli astri che si credevan fissi.
Viaggiatori incerti nel cavo esile della memoria,
dolce è tuttavia sostare laddove il confine è vago
sommersi dall'onda temporale che trascorre.

21

Everyone knows, in one's unquiet soul,
that the beauteous garden is the unfulfilled,
there the vermilion maple-tree grows with the rose,
near to the box-tree the hedged thuja
juts out in full greenery beyond its limits,
the jasmine is enraptured in dimness
on the breaking of the path and there
the faint enclosure yields to the hour of twilight. .

21

Ciascuno sa, nell'animo irrequieto
che il giardino più bello è l'incompiuto,
vi cresce l'acero vermiglio con la rosa,
accanto al bosso la tuia in siepe
sopravanza di verzura i suoi confini,
il gelsomino s'inebria di penombra
laddove il sentiero s'interrompe e
cede il labile recinto all'ora che rabbuia.

Triangles/Triangoli

23

Some among the triangles unmistakably look
vindictive and gloomy- natured.
Revengeful as they are, then, they rage
against the cartographer whose fingers tremble
and anxiety due to the elapsing season
cracks the light on the declining page.
Acute angles are sharply severed across,
more, they are wounded, scratches are pierced
in the unsteady defenceless paper substance.
That is why each and one cartographer's
woeful look happens to oft betray
the shadow of shed tears in his eyes

23

Alcuni tra i triangoli esibiscono
un'indole fosca e assai vendicativa.
Rancorosi, dunque si ribellano
al cartografo cui tremano le dita
e l'ansia della stagione che trascorre
incrina la luce sul foglio che declina.
I lati allora recidono, aspri feriscono

gli angoli acuti, incidono scalfitture
nel supporto cartaceo e indifeso.
Per questo accade che ogni cartografo
mostri spesso nello sguardo triste
l'ombra d'un trattenuto lacrimare. (Fig. 8)

(translated by Claudia Azzola)

8

When a mathematician, about to set off
for a holiday at the seashore
or for calling on some unavoidable relatives,
leaves pages of yet unproven theorems,
resting on white, still untouched papers,
what do the forsaken glasses look at?
will the lenses, without eyes shining,
be bound to magnify delicate golden dust,
sheaves of convergent curved lines
of the neat abandoned theorems?

8

Quando un matematico, in partenza
per destinazione di ferie balneari o
inevitabili visitazioni parentali, lascia
carte di teoremi ancora irresoluti,
posati sui fogli bianchi tuttora intatti,
dove guardano gli occhiali smessi?
resteranno le lenti, senza luce d'occhi,
ormai ad ingrandire pulviscoli dorati,
fasci d'incurvate linee convergenti
dei nitidi teoremi abbandonati?

15

Geometry exhibits shadowy anomalies.
When it simulates the perfection of inviolable certitudes,
it proliferates deceitful and disquieting theorems.
In the unnameable chaos, this precarious order,
brimming with aporias, generates morbid nostalgia
for quadratures and golden albeit impassive sections.
But if shadow escapes all triangulation
and light is the fickle domain of restless winged beings,
perhaps something remains within exiguous enclosures,
sorrowful souls and beloved bodies lightly touching each other,
the transparent fracture of a glance, words engraving
stubborn crevices in the silence and surviving
in the fragile geometry of memory.

15

La geometria mostra ombrose anomalie.
Quando simula perfezione di certezze inalienabili,
prolifera in teoremi ingannevoli e inquietanti.
Nel caos innominabile, quest'ordine precario e
insieme colmo d'aporie, produce insane nostalgie
di quadrature e d'auree ma impassibili sezioni.
Ma se l'ombra sfugge ad ogni triangolazione
e la luce è volubile dominio di volatili irrequieti,
forse entro recinti esigui qualcosa resta,
dolenti anime e corpi amati che si sfiorano,
la limpida frattura d'uno sguardo, parole
che incidono tenaci spiragli nel silenzio e
sopravvivono nell'esile geometria della memoria.

Parallels/Parallele

27

The anxious enigma lies in the axiom
of two lines perfectly defined
in their unquestioned parallel condition.
Geometry, we known, outlines neat certitudes
detached from time and shadows,
but if the meeting point lies at infinity,
the system wobbles, loosens its meshes
and unlocks the unexpected threshold of a foreign,
unexpressed beyond where thought founders.

27

Giace l'ansioso enigma nell'assioma
di due rette perfettamente definite
nell'indubbia loro condizione parallela.
La geometria, si sa, delinea nitide certezze
sottratte al tempo e alla penombra, ma,
posto all'infinito il luogo dell'incontro,
il sistema vacilla, allenta le sue maglie
e schiude imprevista la soglia di un altrove
informulato laddove naufraga il pensiero. (Fig. 7)

(translated by Giuseppe O. Longo)

1

The long side of the rectangle in the shadow does descend.
The short one, in the bare light, wears thin
And vanishes, merely inscribed in the thought
Of an illusory imagined fulfillment.

1

Declina il lato lungo del rettangolo nell'ombra.
Quello breve, nella luce sgombra, s'assottiglia
e svanisce, soltanto inscritto nel pensiero
d'un illusorio compimento immaginato.

Malevich's black square/Il quadrato nero di Malevich

32

As black, in the square looks frightened, yet
settled in its derisory perimeter
prides itself of a vague, rarefied beauty.
Confident in the absolute perfection of its sides
mutely encloses within scant boundaries
than secret, unperturbed pulse
that breezes beyond any known threshold.

32

Nero, il quadrato pare spaurito, ma,
concluso nel suo perimetro irridente
vanta una bellezza vaga, rarefatta.
Certo dell'assoluta perfezione dei suoi lati
muto racchiude entro confini esigui
quel pulsare segreto e imperturbato
che spira oltre ogni soglia conosciuta.

34

The invasion of sense
and the anxiety of any minimal emotions
the suprematist black square repels.
If you look at it, it seem to be loftily staring at you
with Gorgon's square pupil,
as if wanting to tell you in a murmur:
why do you tremble and question
lost within vain enigmas?
The black square IS, nothing it means.

34

Respinge l'invasione del senso
e l'ansia d'ogni minima emozione
il quadrato nero suprematista.
Se lo guardi, pare fissarti altero
con la quadra pupilla di Gorgone,
come volesse dirti sottovoce:
perché tremi e t'interroghi
perduto entro enigmi vani?
Il quadrato nero È, nulla significa.

40

The name, any name, comforts
even the most fearful child,
smoothens frayed drooling at the edges
allows the caressing, lightly touching
in the rustling forest of fhe other names.
But the black square, it has no name
it might be called darkness
that blueish blackening of his,
or sumptuous veil of panther;
though, hearing the whispering
cruel buzzing of a fly
or the echo of scornful laughs, over
his discreet dark threshold
it whisperingly seems to allude
to the light clue of a fault.

40

Il nome, ciascun nome, rassicura
perfino il bimbo più spaurito,
smussa sfrangiate bave ai bordi
permette un carezzevole sfiorarsi
nella selva frusciante d'altri nomi.
Ma non ha nome il quadrato nero,
potrebbe chiamarsi tenebra
quel suo bluastro nereggiare
oppure sontuoso vello di pantera;
sebbene, udendo bisbigliare
il ronzio crudele d'una mosca
o l'eco d'irridenti risatine, oltre
la sua discreta soglia oscura
pare sommessamente alluda
all'indizio lieve d'una colpa.

(translated by Silvio Raffo)

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Part V
Mathematics and Application

À LA RECHERCHE DES RACINES PERDUES (In Search of Lost Roots)

Marco Abate

A task traditionally associated to mathematics is the task of solving equations, and in particular polynomial equations: given a polynomial

$$p(x) = a_d x^d + \cdots + a_1 x + a_0,$$

where d is the *degree* of the polynomial and $a_0, \ldots, a_d$ are the *coefficients* of the polynomial, a *solution* or *root*[1] of the equation $p(x) = 0$ is a number x_c such that the value of the polynomial p computed in x_c is equal to 0.

For polynomials (or, equivalently, equations) of low degree there are formulas for finding the roots starting from the coefficients. For instance, an equation $ax^2 + bx + c = 0$ of degree two has (at most) two roots x_+ and x_- given by the formula

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

and similar (though much more complicated) formulas are known for equations of degree three and four. On the other hand, the Norwegian mathematician Niels Henrik Abel in 1824 has shown that *no such formula can exist for equations of degree five or more*.

But wait a second. Any computer, given the right software (and enough time), is able to find solutions of equations of degree one thousand billions; so, if there are no formulas for solving the equation, how can it do it?

The solution (indeed . . .) of this conundrum lies in what one means by "having a formula". A "formula" is a procedure (an algorithm) that starting from the coefficients of the equation produces roots of the equation. Abel's result says that

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[1] The use of the word "root" to denote a solution of an equation (see, e.g.,[1, p.163]) goes back to the Arabic mathematician al-Khwārizmī, that in 830 AD wrote *Al-jabr,* the first modern algebra book. He thought of the variable x as the hidden source of the equation, as the root is the hidden source of a plant; and finding the solution as akin to bringing to light this source, that is to extracting the root from the soil.

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for equations of degree 5 or more there is no procedure *involving only arithmetic operations and extractions of square, cubic or, more generally, n-th roots* that works for all equations of the same degree; but it does not exclude the existence of other kinds of procedure.

To understand what alternative procedures might exist, let us give another look at the formula for solving equations of second degree. In that formula a square root appears[2]; but how does one compute the square root of a number? Well, there is an algorithm, known since the middle ages (or at least since middle school), yielding arbitrarily good approximations of the square root, up to any prescribed decimal digit. In general this is the most one can hope for: a square root usually is an irrational number, that is a decimal number with infinitely many digits not repeating in a periodic pattern, and thus it cannot be expressed exactly, with all its digits, in a finite amount of time. But for any practical[3] application knowing, say, the first one thousand digits of a square root is more than enough; and, by the same token, knowing the first one thousand digits of the solution of any equation should be enough.

Thus even in the quadratic case "solving" an equation means finding *approximations* of the roots up to a specified precision. This is exactly what (the software in) a computer does: it uses some algorithm for computing approximations to the roots of equations. There are several algorithms available for doing precisely this (see, e.g., [2] for a survey of the more important ones), some of them even faster than the classical formula in the quadratic case. The aim of this short note is to describe (part of) the story of one of the most well-known algorithms, that even though it has been around since the Seventeenth Century it is still being used and studied today, four centuries later: Newton's method.

1600: Newton (et al.)

This algorithm has been developed by Isaac Newton (yes, *the* Isaac Newton) in 1669 in a particular case (and expressed in a somewhat cumbersome way). Shortly later, John Wallis in 1685 and Joseph Raphson in 1690 gave a simpler description of the algorithm; and in 1740 Thomas Simpson devised the modern formulation we shall describe below, and noticed that Newton's method can be also applied to solve some non-polynomial equations (see, e.g., [3] for more details).

The algorithm starts with a *seed*[4], a first guess for the value of a root of the equation, chosen in any way you like; let us call x_0 this guess. The idea is to apply to x_0 a procedure giving a new guess x_1 closer to a root than x_0, and then repeat.

---

2 In a sense, this means that to solve the general quadratic equation is enough to be able to solve the very specific quadratic equation $y^2 = m$.

3 And most unpractical applications too; for the remaining cases usually knowing the exact value of the square root is irrelevant, it suffices to know that it exists (and when it does).

4 Botanical metaphors abound in this subject…

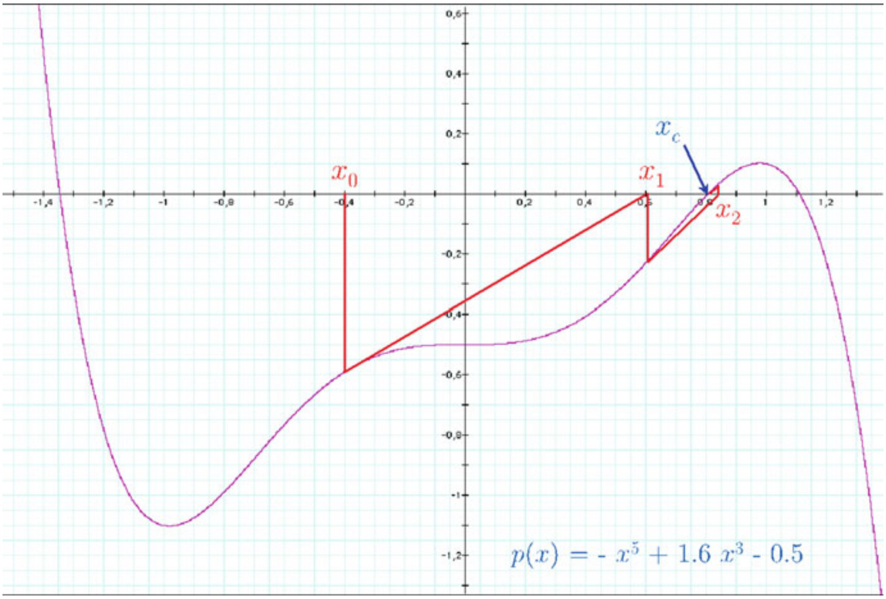

Fig. 1 An example of application of Newton's method

Applying the procedure to x_1 yields a new value x_2 even closer to the root, applying it to x_2 one gets x_3 still closer, and so on; repeating the procedure enough times one gets an approximation of a root as good as needed. It is an example of *recursive* (or *iterative*) algorithm: the same steps are repeated over and over, using as input the output of the previous repetition, until the desired precision is reached.

The procedure for passing from x_0 to x_1 is easily described in graphical terms. Given the polynomial (or more general function) p and the seed x_0, the new approximation x_1 is the x-coordinate of the intersection between the x-axis and the tangent line to the graph of p in the point $(x_0, p(x_0))$. In formula,

$$x_1 = x_0 - \frac{p\,(x_0)}{p'\,(x_0)},$$

where $p'(x_0)$ is the derivative of the polynomial p computed in the point x_0. Figure 1 shows how Newton's method starting from the seed $x_0 = -0.4$ (chosen at random) produces quite a good approximation of a root $x_c = 0.808372\ \ldots$ of the polynomial $p(x) = -x^5 + 1.6x^3 - 0.5$. Indeed, two iterations are enough to get an approximation $x_2 = 0.851758\ldots$ correct up to the first decimal digit, three iterations are enough to get two decimal digits right ($x_3 = 0.80426987\ldots$) and four iterations yield four decimal digits ($x_4 = 0.808349\ldots$).

Of course, to reliably use Newton's method one must show that it works, that is that any sequence x_0, x_1, x_2, $x_3, \ldots$ of numbers obtained in this way actually

gets arbitrarily close (in other words, *converges*) to a root of the original equation. And indeed Newton showed that if x_0 is close enough to a root then the sequence produced by Newton's method does converge to that root, so fast that in general the number of correct decimal digits doubles at each iteration (as happened in the previous example).

So far so good: if the seed is close enough to a root, we have a fast and reliable procedure for finding that root. But wait: if we do not know the value of the root, *how can we possibly choose the seed to be close to a root whose location we do not know?*

1800: Cayley

This is exactly the question posed in 1879 by Arthur Cayley (see [4]), an English mathematician. To try and find an answer to this question, he reformulated it as follows: given a polynomial p, let denote by f the rational function

$$f(x) = x - \frac{p(x)}{p'(x)}.$$

Given a number x_0, put $x_1 = f(x_0)$ and, more generally, $x_n = f(x_{n-1})$ for every natural number $n \geq 1$. In this way starting from the seed x_0 we build a sequence $\{x_n\}$; to be consistent with the botanical terminology used so far we say that x_0 is *sprouting* the sequence $\{x_n\}$. If this sequence converges to a number x_c then necessarily one has $f(x_c) = x_c$, and this can happen if and only if $p(x_c) = 0$, that is if and only if x_c is a root of the polynomial p, as explained by Newton. Therefore the question about when Newton's method works becomes: *for which values of the seed* x_0 *does the sequence* $\{x_n\}$ *converge?* If the call *basin of attraction* of a root x_c the set of seeds sprouting sequences converging to x_c, we would then like to know what is the union of the basins of attractions, and possibly we would like to know something about the geometry of these basins.

Since one has to start somewhere, Cayley began by considering the case of quadratic polynomials. He immediately noticed that the situation became much clearer working with *complex* numbers instead of real numbers, because a quadratic polynomial always has exactly two (possibly coincident) complex roots, whereas it might not have real roots. Thus both the roots and the seed become points in the (complex) plane, and Cayley was able to prove that the bad seeds (that is, the seeds sprouting a sequence not converging to a root) are only the points of the *axis* of the segment connecting the two roots[5]; if the seed is chosen anywhere else Newton's

[5] If the roots are real, the only real bad seed is the middle point of the segment delimited by the two roots. If the polynomial is real but the roots are complex (necessarily conjugated), the axis of the segment connecting the two roots is the real axis; in other words, all real points are bad seeds, consistently with the fact that this polynomial has no real roots.

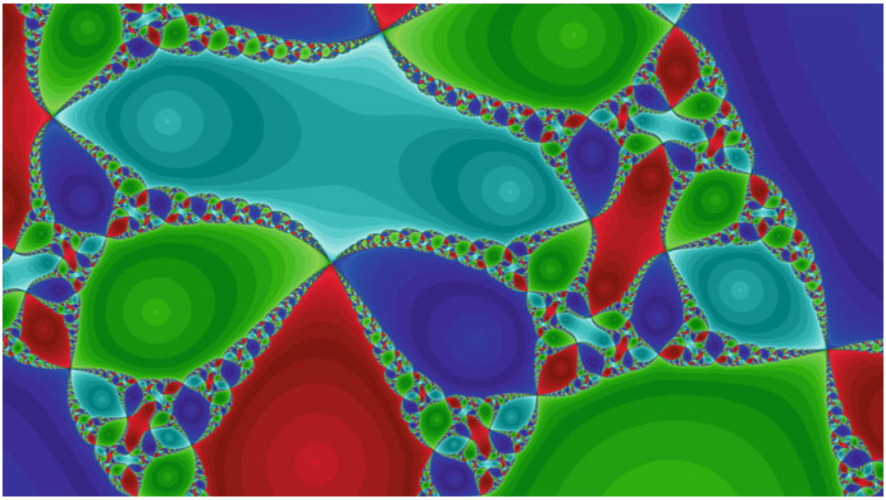

Fig. 2 Basins of attraction of the four roots of a fourth degree polynomial

method works. More precisely, the basin of attraction of a root is the open half-plane (bounded by the axis) containing the root; in other words, if the seed is on the left of the axis then Newton's method yields the left root, if it is on the right then Newton's method yields the right root. As a consequence, since the axis has zero area, choosing a seed at random on the plane we must be very unlucky to pick a point belonging to the axis, and thus applying Newton's method to a random seed will give a root.

So in the quadratic case Newton's method does not work for all seeds, but the set of bad seeds is so small (it has zero area) that if we pick a seed at random it will sprout a sequence converging to a root; in modern mathematical parlance, Newton's method in the quadratic case works *almost surely*. However, the quadratic case is a toy case: one can solve quadratic equations by using the classical formula, Newton's method is not needed. So Cayley tried to understand what happens for cubic polynomials, or for polynomials of higher degree—and he failed miserably. Axes, middle points or any other concept of Euclidean geometry seemed to be utterly useless in this setting. The situation was so complicated that Cayley was even unable to devise sensible questions (let alone answers) about the set of seeds for which Newton's method works . . .

As it became clear a century later, Cayley cannot be blamed for this failure: the mathematical tools needed to deal with the problem were not available yet. And the geometry involved is highly non-trivial, as suggested by Figs. 2 and 3, where different colors identify different basins of attraction; the analogous picture for a quadratic polynomial would have simply shown one half-plane in one color and the other one in a different color.

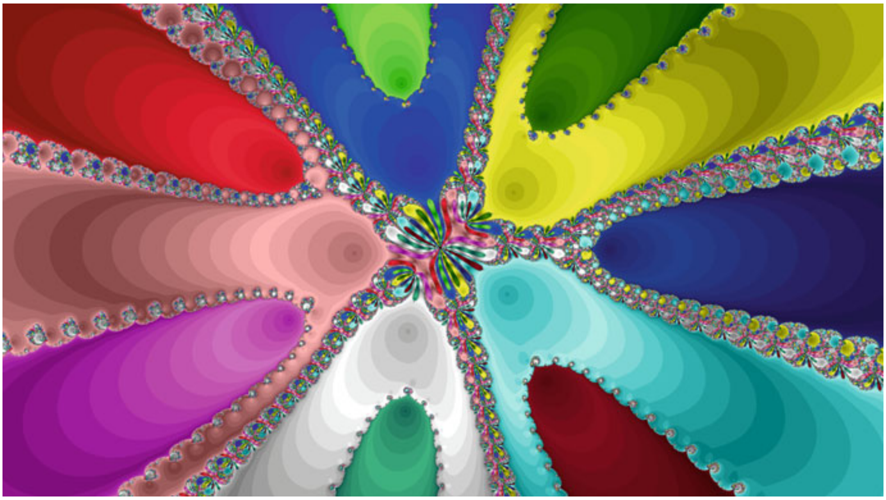

Fig. 3 Basins of attraction of the twelve roots of a twelfth degree polynomial

1900: Fatou

During the latter half of the Nineteenth century and the beginning of the Twentieth century one branch of mathematics flourished (Complex Analysis of functions of one complex variable) and a new one was born (Dynamical Systems). And the deep new tools made available by these two branches allowed a French mathematician, Pierre Fatou, to start understanding the geometry of basins of attractions (see [5–7]).

To give an idea of the kind of results Fatou obtained, let us ask a seemingly unrelated question: is it possible to find in the plane two disjoint (open) sets having the same boundary? Well, that's easy enough: any closed curve without self-intersection subdivides the plane in two sets having the curve as common boundary[6]. For instance, a straight line (e.g., the axis of the segment connecting the two roots of a quadratic polynomial . . .) subdivides the plane in two open half-planes having the line as common boundary.

But what about three (or more) sets? Is it possible to find in the plane three or more disjoint sets all having the same boundary? Apparently not: as soon as we draw any kind of curve on the plane, no matter how short, we get two sides; how can that line be on the boundary of three or more disjoint sets? Clearly it cannot; and thus three disjoint sets with the same boundary cannot exist. Or can they?

The problem in this argument is that it is too naïve: the plane contains sets with boundaries so convoluted that they cannot be considered "curves". Using a terminology that when Fatou was working was yet to be devised, there are sets with

[6] This seemingly self-evident statement actually is a deep theorem of plane geometry known as *Jordan curve theorem;* see, e.g., [8] for a proof.

fractal boundaries. And basins of attraction are a primary source of examples of such sets.

Indeed, the surprising fact proved by Fatou is that *the n basins of attraction of the n roots of a polynomial of degree n all share the same common boundary.* The trick here is that (except in the quadratic case ...) a basin of attraction is not a single piece (is not connected, in modern mathematical terminology), but it is the union of infinitely many pieces all interconnected and touching infinitely many pieces of the other basins of attraction over and over in such a way that from any basin we can cross into any other basin at any point of the boundary.

For instance, around the blue set in the lower center of Fig. 2 one can see a smattering of smaller sets of the other three colors (and some blue too), each one surrounded by even smaller sets of all four colors, and so on ... the net result is that the blue basin (the union of all blue sets) touches the red, green and light blue basins in all points of its boundary, and the same can be said for the other basins. And Fig. 3 shows the same phenomenon with 12 basins ...

A very remarkable fact is that Fatou proved this statement in 1920, when no computer was around and he had no way of drawing, not even sketchily, these sets. He understood what was going on only using abstract mathematical tools, without relying on any kind of geometrical intuition (that furthermore pointed the other way, as we have seen); very remarkable indeed.

Fatou was able to say a lot more on the geometry of the basins of attraction of Newton's method (see, e.g., [9] for a modern survey of his results), but even he was not able to answer the question we are interested in: if we take a seed at random, does the sequence it sprouts converge to a root? In other words, does Newton's method almost surely work for polynomials of arbitrary degree?

2000: Hubbard (et al.)

Somewhat surprisingly, the answer is *no,* and it is so already for cubic polynomials. The first example showing what can go wrong was found in the 1970's, but possibly the easiest one is given by the polynomial $p(x) = x^3 - 2x + 2$. In this case Newton's method is expressed by the rational function

$$f(x) = x - \frac{x^3 - 2x + 2}{3x^2 - 2} = \frac{2x^3 - 2}{3x^2 - 2}.$$

Figure 4 shows (in grey) the three basins of attraction for Newton's methods, and a fourth black region. If the seed belongs to the black region, the sequence sprouted by the seed will eventually land in the central black region, and then it will keep jumping back and forth from that piece to the smaller black piece immediately to its right, without converging to any root (without converging anywhere, actually). For instance, if $x_0 = 0$ we get $x_1 = f(x_0) = 1$, $x_2 = f(x_1) = 0$, $x_3 = f(x_2) = 1$, and so on: the sequence keeps alternating 0 and 1, and neither 0 nor 1 are roots of the

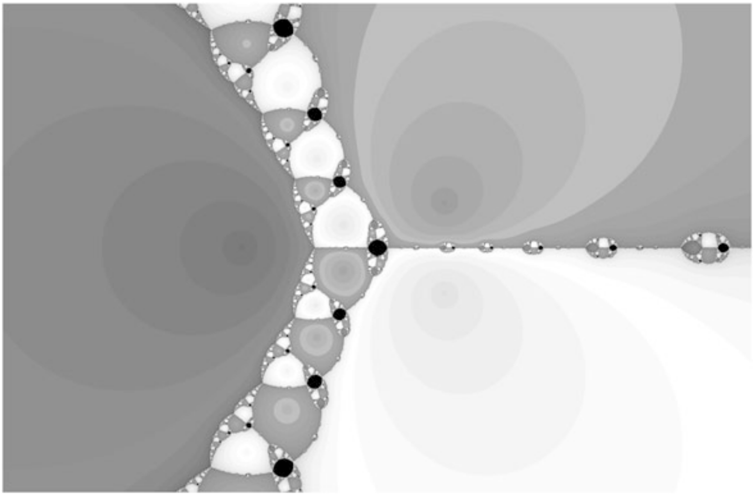

Fig. 4 Basins of attraction of the roots of a cubic polynomial, with a *black region* of bad seeds

polynomial p. Since the black region has a positive area, choosing a seed at random there is a positive probability of picking it in the black region of bad seeds, where Newton's method does not work, and so there is the concrete danger of not being able to find some of the roots of the polynomial p.

Is all lost, then? Should we forget Newton's method and recur to other methods for finding roots of polynomials? Possibly even more surprisingly, John Hubbard, Dierk Schleicher and Scott Sutherland in 2001 have shown (see [10]) that the answer also to this question is *no:* Newton's method *can* be used for finding the roots of polynomials of any degree; it suffices to choose the seeds in a smart way, and not at random.

More precisely, given a degree d, Hubbard, Schleicher and Sutherland have found an explicit way of building a set S_d of points that can be used as seeds for finding *all* roots of *any* polynomial of degree d. In other words, if x_c is a root of a polynomial p of degree d, we can always find in S_d at least one seed x_0 sprouting a sequence converging to x_c.

The construction of the set S_d is a beautiful mixture of geometrical intuition and deep mathematics. Hubbard, Schleicher and Sutherland noticed (and proved!) that all basins of attraction for Newton's methods have "channels" going to infinity, that is strips of a definite width extending to infinity; furthermore, the complicated part of the basins is concentrated only in a bounded part of the plane and along the boundaries of the channels (see for instance Fig. 3, where the channels are clearly visible). Hubbard and collaborators then found a (by no means trivial) way for estimating both the diameter of the complicated part and the width of the channels; and they constructed S_d by picking points placed approximately along a circumference centered at the origin of radius large enough to be outside the

complicated part, and distributed close enough to each other to hit all channels, thus ensuring the presence in S_d of at least one point of each basin.

Another non trivial point here is how many points are in S_d; if S_d contains too many points its use might be unpractical (nobody would try one million seeds for finding the three roots of a cubic polynomial). Now, to find all roots of a polynomial of degree d one clearly needs at least d seeds, one for each root. Finding a set of exactly d seeds able to generate all d roots of any polynomial of degree d might be impossible; but Hubbard and collaborators managed to use only slightly more than d points. Indeed, the set S_d they constructed contains approximately $1.1d(\log d)^2$ points. For instance, S_3 contains 4 points (and not one million), S_4 contains 9 points, S_5 contains 15 points, S_{10} contains 59 punti, S_{50} contains 850 points, and so on.

From the point of view of computer programs, a growth of $d(\log d)^2$ is acceptable; the number of points to deal with is manageable even for polynomials of high degree. Furthermore, in 2012 Schleicher and others [11] have shown that Newton's method applied to S_d is at least as efficient as most of the other known algorithms for finding roots of polynomials.

Summing up, we now have an efficient way of using Newton's method for finding roots of polynomials. But this is just a small part of the story of Newton's method, and there is still a lot to understand and discover about it (see, e.g., [12] for a survey of recent results): after five centuries, there are still beautiful treasures to be found looking for lost roots.

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Probability Calculus, Pitfalls of Intuition and Judicial Errors

Silvia Bozza

Introduction

In the last decades the powerful of the laboratories' analytical results to highlight the dynamic of criminal acts has been object of an increasing attention, partly because of the enormous potentiality of the DNA test that has transformed forensic sciences. Statistics is assuming a fundamental role in forensic science, in particular to quantify the probative value of traces collected at the crime scene, or on individuals suspected to be involved in a crime, and analysed by forensic laboratories. Results of comparative analyses (comparative in the sense that some characteristics of traces recovered at the crime scene are compared to those of material originating from the suspect) are generally presented in numeric form, mostly in probability terms, and need to be carefully interpreted. Moreover, the scientific evidence collected at the crime scene must be considered in conjunction with the bulk of informations the court possesses about the case. Judicial errors correlated to an improper use and interpretation of probability calculus have originated a large debate about (1) how the probative value of the traces is computed, (2) how to interpret the statistical expertise at trial, and (3) how the probative value of the scientific evidence can be combined with the other sources of information (for example, a witness statement) in view of the court decision. Starting from a general description of the potential scenario of interest (section "The Evaluation of Scientific Evidence"), followed by a non exhaustive overview (section "Pitfalls of Intuition") of the possible errors of interpretation that can be originated by intuition which may be a bad substitute of the laws of probability calculus, this paper shows (section "The Transition: Bayes' Theorem") how the Bayes' theorem can represent a formidable tool to combine in a rational way the different sources of information that may become available at trial.

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The Evaluation of Scientific Evidence

Criminal events may leave traces that can be collected by investigators and analyzed by forensic laboratories. In the last years the activities conducted by the police scientifique have gained a major attention, mainly because of the powerful of DNA evidence that is widely used to link suspects to the scene of crime, and whose contribution to the reconstruction and interpretation of facts may be highly relevant. Let us consider the following scenario of interest. A person gains entry to a private house by breaking a window and assaults the owner of the house. The criminal may leave traces of his presence at the crime scene in the form of bloodstains or, say, fibres from his clothings, hairs, fingerprints, shoemarks and so on. Let us imagine that a suspect is apprehended, and that the comparative analyses performed by the forensic laboratories put in evidence similarities amongst the different types of evidence that have been collected. For example, suppose that the crime stains are bloodstains, and that DNA analyses report a 'match' between the genetic profile of the bloodstains collected at the crime scene and the genetic profile of the material originating from suspect. Now, what does these analytical results imply? What is the probative value of a reported match? Is the DNA match a coincidence? Or should we take into account for laboratory's errors? Multiple explanations are possible, and a wider perspective is unavoidable. It must be stressed the attention on the fact that the observation of a match is not incriminating itself, since there may be several different explanations that allow one to justify the laboratories' analytical results. The material at disposal for the analyses may be contaminated, deteriorated, or insufficient, and therefore the analytical results may not always be so easy to interpret. The crime stain might have effectively been left by the suspect, and this will generally be the proposition put forward by the accuse, usually denoted by H_p, where the subscript p denotes the *prosecution*. On the other side, the trace might have been left by a person possessing the same genetic profile of the suspect at the observed *loci*,[1] and this will generally be the proposition put forward by the defence, usually denoted by H_d, where the subscript d denotes the *defence*. The hypotheses of interest to the court may be the following:

H_p: the suspect is the source of the crime stain;
H_d: an unknown person, unrelated to the suspect, is the source of the crime stain.

The question of interest is: what is the contribution of the scientific evidence in support of the competing propositions? In other words, what is the probative value of the evidence? The analytical results performed by the forensic laboratories may certainly represent a fundamental step in the interpretation of facts, nevertheless a court of justice must always deal with uncertainty, even when there are solid judicial elements, since the comprehension of facts happened in the past is by definition incomplete and leaves room for many questions: for example, is the suspect the

[1] A *locus* is a region of DNA where information is collected.

author of the crime? Uncertainty cannot be eliminated, but can be measured and the assessment of the value of the analytical results is in fact generally associated with probabilities as measure of uncertainty. It is crucial that this uncertainty is measured accurately and interpreted correctly to avoid the so-called 'fallacies' or pitfall of intuition [5]. The next section will focus on this aspect.

Pitfalls of Intuition

A large part of the controversy over scientific evidence is due to the way in which the evidence is presented at trial. An historical overview of judicial errors induced by misinterpretations of probability calculus can be found in Aitken and Taroni [1]. It is largely recognized that decision-making is often guided by intuition, and this can also be valid at trial. However, intuition can be a bad substitute for laws of probability in evaluating uncertainty, and the presentation of scientific arguments at trial can create confusion and wrong conclusions. At this regard, there may be found several judicial cases involving conclusions taken on the ground of intuition that turned out to be invalid (see also [2]).

A very common error of interpretation is the *fallacy of the transposed conditional*, also known as *prosecutor's fallacy* and consists in confusing the probability of an hypothesis given the evidence with the probability of the evidence given an hypothesis. Let us denote with the letter I the informations at the court's disposal (say, witnesses, phone calls' printout and so on). On the basis of these informations, the court can quantify the prior probabilities about the competing hypotheses, that are denoted as $Pr(H_p \mid I)$ and $Pr(H_d \mid I)$. Let us suppose that the analytical results reported by the forensic laboratory consist in a 'match', denoted by the letter E, between the compared material, that is between some of the characteristics of the crime stain and those of comparative material originating from the suspect. The court is interested in quantifying the probability that the suspect is the source of the trace (H_p) given that there's a match (E): this can be denoted in probabilistic terms by $Pr(H_p \mid E, I)$. However, the available informations about the frequency of occurrence of a given genetic profile allows one to quantify the probability of the reported match if someone else is at the origin of the trace, that can be denoted by $Pr(E \mid H_d, I)$. In other words, the court will likely ask for an expertise to put light on how the laboratory's analytical results ought to be interpreted. Suppose that the designated expert affirms—on the basis of population studies—that the observed genetic profile is possessed, in mean, by 1 person over 100,000. This is equivalent to quantify in 0.00001 the probability of observing a match (E) if someone else is at the origin of the trace (H_d), that is

$$Pr(E \mid H_d, I) = 0.00001.$$

So, a value is determined for the probability of the evidence if the suspect is innocent. The interpretation of this value can cause large confusion. Let us consider the following possible interpretation:

> *The DNA test is highly relevant. The suspect has the same genetic profile as the attacker. There is a probability equal to* 0.00001 *to observe a match if the suspect is innocent. Therefore, there is a probability equal to* 0.99999 *that the suspect is guilty.*

This interpretation may be tempting and at a first glance it would appear to be a sound conclusion. However, in probabilistic terms this would be equivalent to quantify the probability of guilt as follows:

$$Pr(H_p \mid E, I) = 1 - Pr(H_d \mid E, I) = 1 - 0.00001 = 0.99999.$$

This is clearly wrong, since the court does not know the probability of innocence $Pr(H_d \mid E, I)$: the expertise allows one to compute the probability to observe a match E if a person is innocent, $Pr(E \mid H_d, I)$, and these two quantities are radically different. The first one is the probability of the hypothesis given the evidence, the second one is the probability of the evidence given the hypothesis. The transition from $Pr(E \mid H_d, I)$ to $Pr(H_d \mid E, I)$ can be computed by Bayes' theorem, as it will be shown in the next section.

A second possible error of interpretation is known as *defender's fallacy* [6]. Let us consider again the previous scenario, where a match was reported and the expertise quantified in 1 in 100,000 the frequency of occurrence of the observed genetic profile. The argumentation of a lawyer could be the following:

> *If the conclusions of the expert are correct, then in a city like the one in which the crime occurred, with a population of 200000 people who may have committed the crime, the same genetic profile would be found in, approximately, 2 people. The evidence merely shows that the suspect is one of two people, and the DNA test has provided a probability of guilt of 1 in 2. Such a probability is too small and has little relevance for proving the suspect is guilty.*

This point of view is partially correct: the probability of guilt, on the basis of the reported match is 1 in 2. The error lies in the second argumentation concerning the relevance of the evidence, but this will be discussed in the next section.

Another error that may affect the computation of the probative value of the evidence is the *error of dependent events*, that consists in assuming independence among events when this is not reasonable. As an example, let us consider the case of Sally Clark, who was convicted for the murder of her two infants. The first death, at the age of 3 months, was initially treated as a case of sudden infant death syndrome (SIDS), but after the death of her second child in similar circumstances she was arrested and charged for murdering both her children. At trial, a professor of pediatrics, quoted that in a family like the Clarks, the probability that two babies would die of SIDS was around 1 in 73 million. The witness went to square the estimate of the frequency of SIDS in families having some of the characteristics of the Clarks, that was estimated around 1 in 8,500. This calculation has been shown being invalid [3]. It would only be valid if SIDS cases arose independently

between cases, but there are very strong reasons for supposing that it is not the case. There may be in fact genetic or environmental factors that predispose families to this syndrome, so that a second case of SIDS becomes more likely. Other experts quantified in 1 in 60 the probability of a second case, producing a considerable decrement of the previous probability to 1 in 130,000. This value is still quite small and could be easily misinterpreted, as it will be underlined in the next section.

The Transition: Bayes' Theorem

The court, given the informations about the case and the scientific evidence wants to quantify the probability of guilt $Pr(H_p \mid E, I)$. However, starting from the laboratory's analytical results it may be possible to compute the probability of the evidence given the hypothesis H_d, that is $Pr(E \mid H_d, I)$. The transition can be performed by applying the Bayes' theorem, that allows one to compute, respectively, the posterior probability of guilt

$$Pr(H_p \mid E, I) = \frac{Pr(E \mid H_p, I)Pr(H_p \mid I)}{Pr(E \mid H_p, I)Pr(H_p \mid I) + Pr(E \mid H_d, I)Pr(H_d \mid I)}, \tag{1}$$

and the posterior probability of innocence

$$Pr(H_d \mid E, I) = \frac{Pr(E \mid H_d, I)Pr(H_d \mid I)}{Pr(E \mid H_p, I)Pr(H_p \mid I) + Pr(E \mid H_d, I)Pr(H_d \mid I)}. \tag{2}$$

Dividing Eq. (1) by Eq. (2), one obtains the *odds-form* of Bayes theorem, that represents a very useful result for forensic sciences,

$$\frac{Pr(H_p \mid E, I)}{Pr(H_d \mid E, I)} = \frac{Pr(E \mid H_p, I)}{Pr(E \mid H_d, I)} \times \frac{Pr(H_p \mid I)}{Pr(H_d \mid I)}. \tag{3}$$

Let us examine the three ratios in Eq. (3). The last ratio at the right-hand side, $Pr(H_p \mid I)/Pr(H_d \mid I)$, represents the *prior odds* against the defence: a value greater (lower) than 1 supports the prosecution's (defence's) hypothesis. The ratio at the left-hand side, $Pr(H_p \mid E, I)/Pr(H_d \mid E, I)$, represents the *posterior odds* against the defence and encapsulates the prior informations at the court's disposal and the scientific evidence: again, a value greater (lower) than 1 supports the prosecution's (defence's) hypothesis. The first ratio at the right-hand side, $Pr(E \mid H_p, I)/Pr(E \mid H_d, I)$, represents the *likelihood ratio* and is denoted by the letter V: it is the ratio of the probabilities of the evidence assuming guilt (numerator) and innocence (denominator) of the suspect. The likelihood ratio V allows one to compute the value of the evidence, and provides an expression of how the evidence is capable of discriminating amongst the competing propositions at trial. The interpretation of the

scientific evidence can be seen as a comparison between two probabilities [4]: the probability of the evidence assuming the proposition put forward by the prosecution to be true (numerator), and the proposition of the evidence assuming the proposition put forward by the defence to be true (denominator), that is

$$V = \frac{Pr(E \mid H_p, I)}{Pr(E \mid H_d, I)}.$$

This implies that it is not enough for the accuse to show that the evidence is unlikely if the suspect is innocent: the evidence has also to be more likely if the suspect is guilty. Consider the Sally Clark's conviction cited in the previous section. The reported probability of 1 in 73 million, leaving aside the matter of validity, may also be easily misinterpreted. At the time, some press reports stated that this was the chance that the deaths of Sally Clark's children were accidental: if this conclusion was valid, then the probability of guilt would be extremely large and equal to 1- (1 in 73 million). However, this conclusion is wrong, and represent a prosecutor's fallacy. Two competing explanations must be taken into account by the jury: the fact that two deaths by SIDS is quite unlikely is of little value if taken alone, since two deaths by murder may well be even more unlikely [3]. In the same way, it is not correct to consider only the probability of the evidence if the suspect is guilty and to conclude that a high value is indicative of guilt: the evidence may be likely also in the case the suspect is innocent.

The odds form of Bayes' theorem represents a very important result for forensic sciences: the court will focus on questions such as *'what is the probability of the hypotheses of interest given the evidence?'* (posterior odds), while the expert will focus on questions such as *'how much the evidence supports the prosecutor's and the defences's hypotheses?'* (likelihood ratio). The prior odds in favor of guilt can be updated to the posterior odds given the evidence E. This can easily be done by multiplying the prior odds by the likelihood ratio, as in Eq. (3). Let's go back to the defender's fallacy presented in section in "Pitfalls of Intuition", and to the defender's point of view according to which the probability of guilt after the DNA reported match is too small and has little relevance for proving the suspect is guilty. The likelihood ratio for this case can be easily shown being equal to 100,000. The numerator $Pr(E \mid H_p, I)$ can be taken equal to 1 (under the assumption that no laboratory's errors have been committed, if the suspect is at the origin of the trace, the laboratory will correctly report a match), while the denominator $Pr(E \mid H_d, I)$ is given by 1/100000. Technically, the lawyer's point of view is correct: the probability of guilt is 1 in 2. However, before the evidence of the DNA test was available, the suspect had a probability of 1 in 200,000 of being guilty (a part any other evidence that may be presented). The effect of the evidence is to increase this probability by a factor of 100,000. The evidence is 100,000 more likely if the suspect is guilt than if he is not guilt. So, the evidence may be quite compelling in support of an hypothesis of guilt, though of course this is unlikely to be sufficient for a verdict of guilty.

Conclusion

Criminal events may leave traces of different type that can be collected by investigators and analyzed. Even when solid judicial elements are available, the reconstruction of facts is unavoidably characterized by uncertainty, since there may be different explanations for the observed evidence. The assessment of the value of the analytical results is associated with probabilities as measures of uncertainty, and it is important to ensure that this uncertainty is measured accurately and represented correctly, to avoid misinterpretations that may give rise to severe judicial errors. Statistical reasoning represent a fundamental tool that may be of valuable help both for forensic scientists and for jury to take terminal decisions.

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Photos, Objects and Computer Vision

Maurizio Falcone and Alessandra Seghini

In his novel "Let's stay together" of 1956, Isaac Asimov has described a critical situation: an officer of the US was interviewing the head of the Department of Robotics in Cheyenne to know more about the research in that area. He was scared by the fact that ten humanoids built by the enemy where walking around the States. At that time this was completely unrealistic, but it seems that the times have changed.

Nowadays robots can walk, read, speak, play a piano and behave as humans in many circumstances although their "intelligence" is still rather limited. However, recent advancements in computer vision can make them recognize a scene or an object and walk safely in an unknown environment avoiding obstacles. This is due to the findings in many research areas, including computer vision. Here we will describe in particular the Shape-from-Shading (SfS) problem, an inverse problem which amounts to reconstruct the 3D structure of an object given one 2D grey value image. For this task, the SfS process relies on informations on the illumination and the light reflectance in the scene. This problem, first introduced by Horn [9] in his PhD thesis, is a classic inverse problem in computer vision with many potential applications beyond robotics, see e.g. [6, 7, 11, 16, 20].

The Classical Shape from Shading Model

Let us give a brief outline of the SFS problem and introduce the basic assumptions for the classical model based on a single image. We attach to the camera a three-dimensional coordinate system ($Oxyz$), such that Oxy coincides with the image plane and Oz coincides with the optical axis, orthogonal projection is usually considered to simplify the model. Under this simplification, the visible part of the scene is, up to a scale factor, a graph $z = u(\mathbf{x})$, where $\mathbf{x} = (x, y)$ is an image point. The datum in this problem is our image which is represented by the *irradiance (or brightness)*

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function I. The function I is measured at each point (pixel) of the image, for example in terms of a greylevel (from 0 to 255). However, to construct a continuous model, we will assume that I takes real values in the interval [0, 1] where 0 corresponds to black and 1 to white passing through infinitely many gray levels. The SFS problem is characterized by the *image irradiance equation*:

$$R(\mathbf{n}(\mathbf{x})) = I(\mathbf{x}), \tag{1}$$

where $I(\mathbf{x})$ is the irradiance (or brightness) measured in the image at point $\mathbf{x}$ and $R(\mathbf{n}(\mathbf{x}))$ is the reflectance function, which gives the value of the light re-emitted by the surface as a function of its orientation i.e., of the unit normal $\mathbf{n}(\mathbf{x})$ to the surface at the point $(\mathbf{x}, u(\mathbf{x}))$. This normal can be expressed as:

$$\mathbf{n}(\mathbf{x}) = \frac{1}{\sqrt{1 + p(\mathbf{x})^2 + q(\mathbf{x})^2}} (-p(\mathbf{x}), -q(\mathbf{x}), 1), \tag{2}$$

where

$$p = \partial u/\partial x, \; q = \partial u/\partial y \text{ so that } \nabla u(\mathbf{x}) = (p(\mathbf{x}), q(\mathbf{x})) \tag{3}$$

(here $\partial u/\partial x$ and $\partial u/\partial y$ are the partial derivatives uf u with respect to the x and y coordinates and ∇u is the gradient of u). The height function u, which is the unknown of the problem, has to be reconstructed on a compact domain $\Omega \subset \mathbb{R}^2$, called the "reconstruction domain". Assume that there is a unique light source at infinity whose direction is indicated by the unit vector $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$. and let us assume, for simplicity, that ω is given. In order to simplify the model, it is usually assumed that the surface has uniform reflectance properties, so the light which is reflected by the surface just depends on the angle between the normal to the surface and the direction of the light ω scaled by a positive factor γ (the albedo). A surface with these properties is called a *Lambertian surface* and for it we can finally write (1) as $R(\mathbf{n}(\mathbf{x})) = \gamma\omega \cdot \mathbf{n}(\mathbf{x})$. Setting the light source at infinity guarantees that all the light rays are parallel. Then, setting the uniform albedo equal to 1 and applying definition (2), we obtain

$$I(\mathbf{x})\sqrt{1 + |\nabla u(\mathbf{x})|^2} + (\omega_1, \omega_2) \cdot \nabla u(\mathbf{x}) - \omega_3 = 0 \qquad \text{for } \mathbf{x} \in \Omega, \tag{4}$$

which is a first order non-linear partial differential equation of the Hamilton-Jacobi type (here $\cdot$ is the scalar product). We will also consider the simplified equation which appears in most of the literature and corresponds to a vertical light source at infinity i.e., $\omega = (0, 0, 1)$. Then (4) reduces to the *eikonal equation*

$$|\nabla u(\mathbf{x})| = f(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega, \tag{5}$$

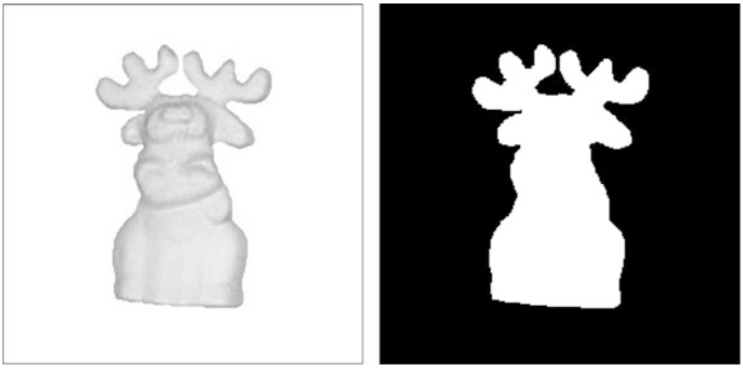


Fig. 1 Object with occluding boundary, which might be used to define the silhouette Ω (the *white region* on the *right*)

where

$$f(\mathbf{x}) = \sqrt{\frac{1}{I(\mathbf{x})^2} - 1}. \tag{6}$$

Since this equation just depend on ∇u, if u is a solution also $u+constant$ is a solution so one has (at least) to add the value at the boundary of the silhouette Ω in order select a unique solution (we will see later that this is not enough to get uniqueness). Equation (4) or (5) must be complemented with boundary conditions on $\partial\Omega$ and with additional information to select a unique solution. For an image containing an "occluding boundary", it is usual to consider this boundary as $\partial\Omega$. For example, in Fig. 1, the part of the image representing the object in greylevels is the *silhouette* is Ω and $\partial\Omega$ coincides with the occluding boundary.

A natural choice is to consider Dirichlet type boundary conditions in order to take into account (at least) two different possibilities. The first corresponds to the assumption that the surface is standing on a flat background i.e., we set:

$$u(\mathbf{x}) = 0 \qquad \text{for } \mathbf{x} \in \partial\Omega. \tag{7}$$

The second possibility occurs when the height of the surface on the occluding boundary is known, e.g. this happens when we know (or assume) that the object is a surface of revolution around a given axis. This situation leads to the more general condition:

$$u(\mathbf{x}) = g(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \partial\Omega. \tag{8}$$

The solution of the above Dirichlet problems (4)–(7) or (4)–(8) will characterize a surface corresponding to the brightness function $I(\mathbf{x})$ measured in Ω which is the

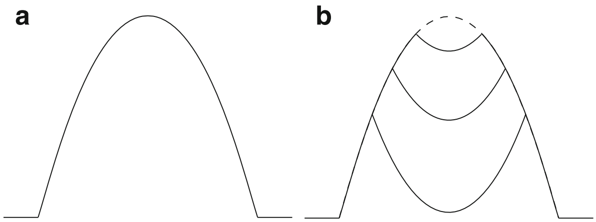


Fig. 2 Illustration of the concave/convex ambiguity: (**a**) maximal solution and (**b**) a.e. solutions giving the same image

datum of the problem. Points $\mathbf{x} \in \Omega$ such that $I(\mathbf{x})$ is maximal correspond to the particular situation where ω and $\mathbf{n}(\mathbf{x})$ point in the same direction: these points are usually called "singular points" and they play an important role to characterize a unique solution.

The first remark is that nonlinear partial differential equation such as (1) do not have in general regular solutions (i.e. solutions which are differentiable at every point). So the natural framework to set the problem is that of weak solutions which do not satisfy the equation everywhere but are at least continuous with (possibly) discontinuities in the gradient. Starting from the paper by Rouy and Tourin [18], the most recent approach to the resolution of SFS uses the notion of *viscosity solutions* to first order PDEs, see e.g. [1]. This is also natural from the point of view of computer vision applications: if you look around you can see many objects whose surface is not smooth. Unfortunately, the Dirichlet problem (4)–(8) can have several "weak solutions" in the viscosity sense and also several classical solutions. As an example, all the surfaces represented in Fig. 2 are viscosity solutions of the same equations (5)–(7), which is a particular case of (4)–(8). The solution represented in Fig. 2a is the maximal solution and is regular (differentiable). All the non-smooth a.e. solutions which can be obtained by a reflection with respect to a horizontal axis, are still admissible weak solutions (see Fig. 2b). In fact the value of the scalar product $\omega \cdot \mathbf{n}(\mathbf{x})$ corresponding to any of these surfaces is the same for every $\mathbf{x}$ and coincides with the same value computed on the regular solution (Fig. 2a). This is the *concave/convex ambiguity* of the SfS model. In this example, the lack of uniqueness of the viscosity solutions is due to the presence of a singular point, where the right hand side of (5) vanishes. An additional effort is then needed to define our solution since the lack of uniqueness is also a big drawback when trying to compute a numerical solution. In order to circumvent those difficulties, the problem is usually solved by adding some informations such as the height at each singular point [14] or the complete knowledge of a level curve [12, 13]. More recently, an attempt has been made to eliminate the need for a priori additional information. Always in the framework of viscosity solutions an effort has been made to characterize the *maximal solution* without additional informations and to build approximation schemes which converge to that solution [3, 8]. Let us finally mention that Eq. (4) is not the most general equation for the SfS. We made along

the way many assumptions but, in general, real materials are not Lambertian, the albedo is not constant, the direction ω is not known a priori. The situation is even more complex if interreflections are taken into account, but will not treat these cases in detail.

In order to obtain a *numerical solution of the SfS problem* two main strategies are available. The first is to solve directly the partial differential equation combined with the boundary conditions, this method will give an approximation of u provided additional conditions on the approximation scheme are satisfied. The second is the *variational approach* where we work first on an optimization problem in the unknown (p, q) at every pixel of the image. Then, using the field (p, q) and recalling their definition (3) we can reconstruct u by an integration along a path, made for example by a an horizontal plus a vertical step. We will not give the details here but they can be found in the survey [7].

The first approach is rather standard and the difficulty is to make the discretization of the continuous equation converge to the right solution. In the variational approach the main point is the choice of the functional to be minimized in the first step. In this approach, some regularity of the surface is assumed, and the approximate solutions computed are typically local minima of the functional. But u appears, in the image irradiance equation, only through its first derivatives p and q, which are two non-independent functions. In fact, if u is regular and twice differentiable

$$\partial p/\partial y = \partial q/\partial x. \tag{9}$$

and this condition guarantees integrability along a path. The only problem with these unknowns is that p or q become infinite at every point $\mathbf{x}$ belonging to the occluding boundary. This problem is not a cause for concern if no point $\mathbf{x}$ in the reconstruction domain Ω is such that $I(\mathbf{x}) = 0$. As Eq. (9) is a hard constraint on p and q, the most natural functional associated with Eqs. (1) and (9) is

$$\begin{aligned}\mathscr{F}_1\,(p, q, \mu) = &\int_{\mathbf{x}\in\Omega} [r(p(\mathbf{x}), q(\mathbf{x})) - I(\mathbf{x})]^2\; d\mathbf{x}\\ &+ \int_{\mathbf{x}\in\Omega} \mu(\mathbf{x})\,[\partial p/\partial y(\mathbf{x}) - \partial q/\partial x(\mathbf{x})]\; d\mathbf{x},\end{aligned} \tag{10}$$

(here the function r is such that $R(\mathbf{n}(\mathbf{x})) = r(p(\mathbf{x}), q(\mathbf{x})))$ Since the minimization of $\mathscr{F}_1$ is rather slow, Horn and Brooks have also proposed the functional,

$$\begin{aligned}\mathscr{F}_2\,(p, q) = &\int_{\mathbf{x}\in\Omega} [r(p(\mathbf{x}), q(\mathbf{x})) - I(\mathbf{x})]^2\; d\mathbf{x}\\ &+\lambda_{\text{int}} \int_{\mathbf{x}\in\Omega} [\partial p/\partial y(\mathbf{x}) - \partial q/\partial x(\mathbf{x})]^2\; d\mathbf{x}.\end{aligned} \tag{11}$$

where the constraint term for integrability becomes a penalty term and the positive constant weight λ_{int} called "integrability factor" (note that the choice of λ_{int} is arbitrary).

The PSFS Model and Related Equations

Several improvements with respect to the original problem presented in the previous section have been studied. One of the most relevant is the introduction of perspective deformations in the model as proposed in [5, 16, 19], also taking into account a light attenuation factor [17].

Let us sketch the model for PSFS with point light source located at the optical center O and light attenuation term (see Fig. 3). Let $\mathbf{x} = (x, y)$ be a point in the image domain Ω, where Ω is an open bounded subset of $\mathbb{R}^2$. Let $I = I(x, y) > 0$ be the normalised brightness function. We have $I = \frac{E(x,y)}{\sigma}$, where E is the greylevel of the given image and σ is the product of the surface albedo (which tells us to which extent the surface reflects light) and the light source intensity. Moreover, f denotes the focal length, i.e. the distance between the optical center C of the camera and the two-dimensional plane to which the scene of interest is mapped.

M will represent a generic point on the surface Σ to be reconstructed. We choose as unknown of the problem the function $u : \Omega \to \mathbb{R}$ such that

$$M = M(x, y) = u(x, y)\, m' , \tag{12}$$

where

$$m' = \frac{\mathsf{f}}{\sqrt{x^2 + y^2 + \mathsf{f}^2}}\, m \quad \text{and} \quad m = (x, y, -\mathsf{f})^\top . \tag{13}$$

Note that, according to these notations, $u > 0$ holds as the depicted scene is in front of the camera. Here we denote by $r(x, y)$ the distance between the point light source and the point $M(x, y)$ on the surface. It holds $u(x, y) = r(x, y)/\mathsf{f}$, since the light source location coincides with the optical center.

The model associated to the PSFS problem is obtained again by the image irradiance equation (1)

We denote by $\omega(x, y)$ the unit vector representing the light source direction at the point $M(x, y)$ (note that in the classic SFS model this vector is constant but this is not the case here):

$$\omega(x, y) = \frac{(-x, -y, \mathsf{f})^\top}{\sqrt{x^2 + y^2 + \mathsf{f}^2}} . \tag{14}$$

Adding the assumptions of a *light attenuation term* and of a *Lambertian surface*, the function R is defined as

$$R(\mathbf{n}(x, y)) = \frac{\omega(x, y) \cdot \mathbf{n}(x, y)}{r(x, y)^2} , \tag{15}$$

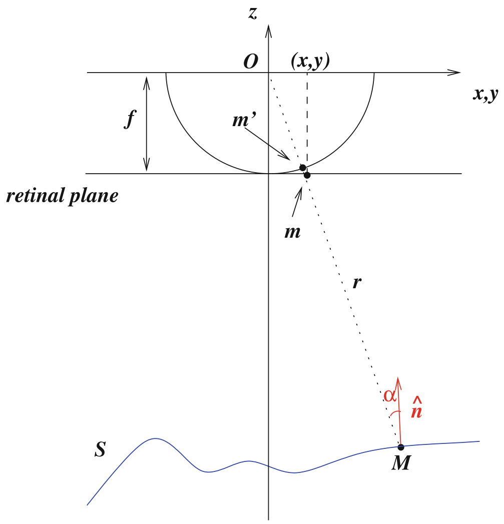


Fig. 3 Notations for the PSFS model with point light source at optical center O

with an attenuation factor which is equal to the inverse of the squared distance from the source. Expression (15) would still hold for any location of the point light source, but the same would not be true for the equality $u(x, y) = r(x, y)/\mathsf{f}$ nor for (14). The case of a light source coinciding with the optical center corresponds more or less to endoscopic images [16] and to photographs taken at short distance with the camera flash [17]. Another notable advantage of the PSFS model using a point light source at the optical center is that there is no shadow in the image.

Finally, by the irradiance equation and (15) we obtain the PSFS equation

$$\frac{\omega(x, y) \cdot \mathbf{n}(x, y)}{r(x, y)^2} = I(x, y). \tag{16}$$

In order to write down the corresponding PDE, it is useful to introduce the new unknown $v = \ln(u)$ (we recall that $u > 0$). Finally, Eq. (16) can be written as

$$H(x, y, v, \nabla v) := \frac{I(x, y)}{Q(x, y)} \mathsf{f}^2\, W(x, y, \nabla v) - e^{-2v(x,y)} = 0\,, \quad (x, y) \in \Omega \tag{17}$$

where

$$Q(x, y) := \frac{\mathsf{f}}{\sqrt{x^2 + y^2 + \mathsf{f}^2}} \tag{18}$$

and

$$W(x, y, \nabla v) := \sqrt{\mathsf{f}^2 \|\nabla v\|^2 + (\nabla v \cdot (x, y))^2 + Q(x, y)^2}, \tag{19}$$

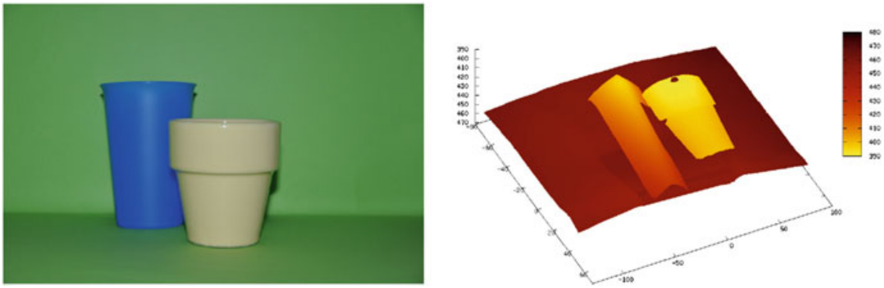


Fig. 4 (*Left*) Real image; (*right*) 3D reconstruction

(here $\|\cdot\|$ denotes the Euclidean norm). Figure 4 shows the result for this model on a real image representing two cups (see [2] for more details on these models).

Recent Developments on Shape from Shading Models

More recently the classic model based on Lambertian assumptions and on orthographic projection has been improved in order to deal with more realistic assumption and improve the results on real images. For example, one can consider the light source at finite distance from the object so that the rays are not parallel and/or include black shadows (i.e. regions where $I(x) = 0$) in the image. The black shadows are usually forbidden for two main reasons. The first reason is technical, in the classical model the eikonal equation can be applied only where $I(x) > 0$ because I appears at the denominator of the right-hand side term (6). The second is the remark that if the image is black we do not have informations on the real surface so we can not reconstruct it. Every surface whose height is below the "black ray" separating light and shadow could be a reasonable solution. Note that "black shadow" regions are typically projected by the object on the background and/or on its own surface and appear when the light source moves away from the vertical. This situation often appears in real images taken at sunset or at dawn. Since it is impossible, for lack of information, to reconstruct the real surface corresponding to these regions we can follow two different strategies. Either we apply the model only to the regions where $I > 0$ and deal with all the boundary conditions corresponding to these regions, or we solve the problem on the whole image and reconstruct a "virtual surface" on the black shadow regions. It is reasonable to define this virtual surface as the one described by the rays separating light and shadows.

The model has been extended in order to include in the solution the flat surface separating light and shadow (a result is shown is Fig. 5).

In Stereo Shape from Shading, one can consider two images *from different point of view under the same light conditions*. This situation mimics our vision system

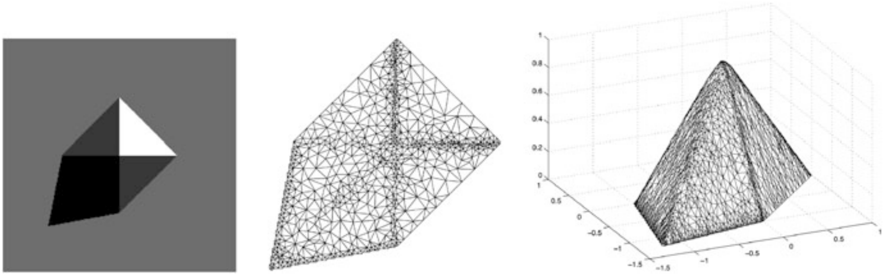


Fig. 5 From *left to right*: the real image with black shadows, the grid discretization of the domain Ω, the 3D reconstruction

based on two eyes separated by a small distance. It has been shown that for this specific problem there is just one solution and the problem is well-posed [4].

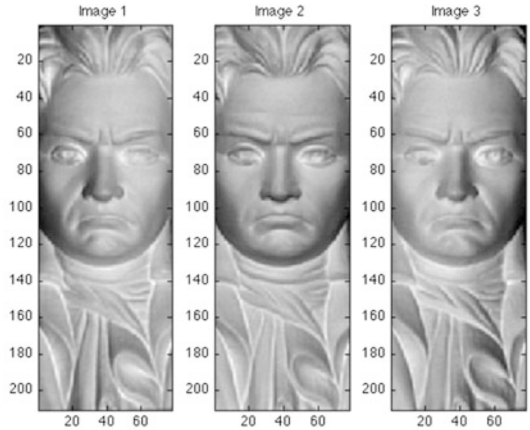
Another, recent improvement is the *stereophotometric approach* which is now considered a very promising technique. In this approach, we use several images *from the same point of view but taken under different light source directions*. This is the laboratory situations where one can move the light source around an object while taking several pictures. Clearly, for every image we can use one of the previous models for SfS (orthographic, perspective, light attenuation) and this gives rise to a system of nonlinear partial differential equations. For example, by applying the orthopraphic model and taking two images I_1 and I_2 which correspond respectively to two different light source directions ω^1 and ω^1, we get the following system of nonlinear partial differential equations

$$\begin{cases} I_1(\mathbf{x})\sqrt{1+|\nabla u(\mathbf{x})|^2} + (\omega_1^1, \omega_2^1)\cdot\nabla u(\mathbf{x}) - \omega_3^1 = 0, \text{ for } \mathbf{x} \in \Omega \\ I_2(\mathbf{x})\sqrt{1+|\nabla u(\mathbf{x})|^2} + (\omega_1^2, \omega_2^2)\cdot\nabla u(\mathbf{x}) - \omega_3^2 = 0, \text{ for } \mathbf{x} \in \Omega \end{cases} \tag{20}$$

Note that the number of equations is determined by the number of images and in some cases the system has a unique solution with just two images although, in general, to select a unique solution three images are necessary. For example, starting from the three images of the Beethoven bust (Fig. 6) we can compute the solution of the system and get back the 3D reconstruction of the surface (Fig. 7) (see [15] for more details).

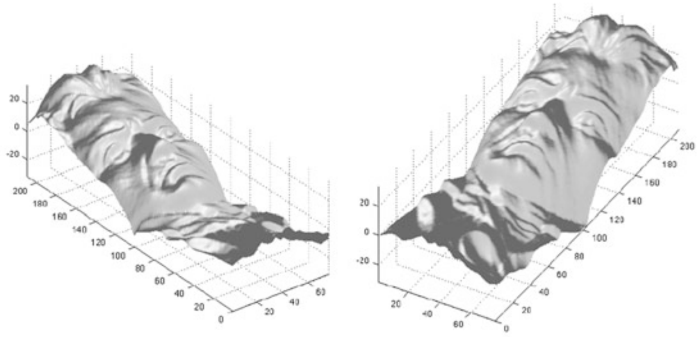
As we mentioned at the beginning, the above techniques allow for a rather accurate 3D reconstructions of the objects appearing in a scene. Then, this technology can be applied in medical surgery where the image acquisition is obtained via an endoscope and is important to have a precise description of the area. At a macroscopic level the same inverse problem appears in space investigations when a satellite takes several pictures of the same area allowing for the construction of its altitude map which can be used later for robot explorations in the area.

Another interesting applications of the SfS models is related to the preservation of cultural heritage. The advanced techniques for 3D reconstruction are based on



$$I_1 \text{ for } \omega' : \varphi_1 = 0.263, \theta_1 = -0.305 \quad I_2 \text{ for } \omega'' : \varphi_2 = 0.2, \theta_2 = 3.226$$

Fig. 6 Photometric stereo: Three real images of Beethoven bust



$$I_1 \text{ for } \omega' : \varphi_1 = 0.263, \theta_1 = -0.305 \quad I_2 \text{ for } \omega'' : \varphi_2 = 0.2, \theta_2 = 3.226$$

Fig. 7 The 3D reconstruction of the Beethoven bust starting from the data in Fig. 6

laser or optical measurements but 3D scanners are rather expensive since a specific hardware for the data acquisition is required (see for example the movie presentation [21]). An accurate reconstruction based on SfS model could produce a digitization of a statue just using low cost cameras in a controlled environment where one can adjust the light sources. This is clearly an advantage since the method can be easily applied with low investments.

Another example comes from the digitization of ancient manuscripts. These books are very delicate and their reproduction requires an expensive hardware in order to preserve the volumes. Moreover, the copy of the manuscript via a standard camera has a big perspective distortion which is due to the curvature of the pages

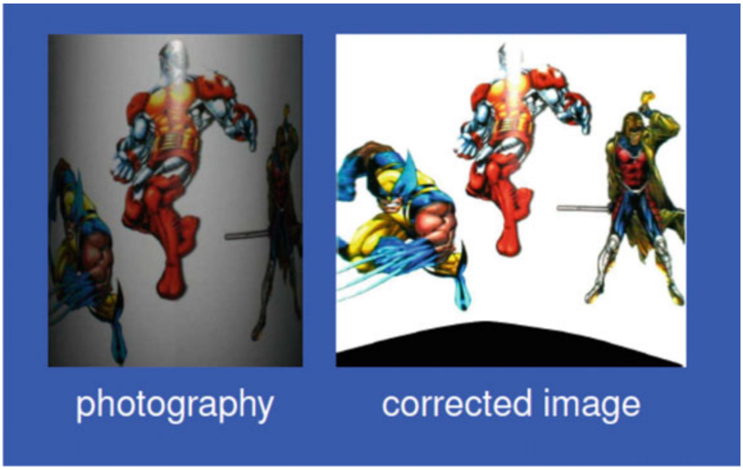


Fig. 8 An example of corrections for book reproduction via SfS models

near the center of the book. Typically the text is not horizontal due to this distortion and classical OCR (Optical Character Recognition) systems cannot work on it, so an automatic classification or analysis of the text is not feasible. Via SfS technique one can compute the real surface of the pages and then reapply the content of the page on it correcting the perspective distortion as in Fig. 8 (more details can be found in [6]).

So computer vision can really help robots to make a walk and, finally, the science fiction novel imagined by Asimov becomes reality.

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Sandpiles and Earthquakes

Gian Marco Todesco

Introduction

This aerial photo of the San Andreas Fault is impressive (Fig. 1). The fault extends itself over roughly 1,300 km through California in the United States. It follows the boundary between the Pacific Plate and the North American Plate. The two tectonic plates move against each other (only few centimeters per year) slowly but firmly. Their motion affects rocks of the earth crust and deforms them elastically. Eventually, elastic stress gets so high that it exceeds rock strength, causes fractures and releases most of the accumulated energy.

When seismic waves produced in this event reach the earth surface, they generate vibrations that constitute earthquakes. Indeed most earthquakes (but the deepest ones) occur close to faults. Many faults are underground and sometimes they can be recognized only because of the generated quakes, whereas the San Andreas Fault is a large structure very close to the surface and therefore the fracture is clearly visible. The surrounding area has been struck by a number of notable earthquakes since historic times; for instance the San Francisco earthquake of 1906 is one of the worst natural disasters in the history of the United States and was felt as far away as Oregon and central Nevada.

In the 1980s, the connection between fault movements and earthquakes was well understood. Moreover records in the area showed an evident regularity in the occurrences of significant earthquakes: six large earthquakes, with similar features, happened roughly every 20 years from 1857 to 1966. This led in 1984 to predicting a similar event around 1993. The United States Geological Survey (USGS) conducted an unprecedented experiment and installed an elaborated array of instruments around Parkfield, a city near the fault, to witness the earthquake and identify any precursors when the quake would come. It has been probably the most heralded scientific earthquake prediction ever. Eventually the earthquake came, but only in 2004 (a decade later).

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Fig. 1 "The San Andreas Fault, on the Carrizo Plain", by Doc Searls, http://www.flickr.com/photos/docsearls/15392616/. Licence at http://creativecommons.org/licenses/by-sa/2.0/

This anecdote illustrates well the difficulty in predicting earthquakes. Today we know a lot of things about them but still we are not able to exactly predict a specific event. Charles Richter, the prominent seismologist most famous for creating the Richter magnitude scale that quantifies earthquakes size, once said: "Only fools and charlatans predict earthquakes".

Big earthquakes, however, have such catastrophic effects that immense efforts are put in trying to better understand the phenomenon. While single events are elusive, the distribution of events follows rather well some statistical laws that can be studied in depth.

One of the most important is the Gutenberg-Richter law, first introduced in 1942. This law correlates size (measured in *magnitude*) and frequency of earthquakes. As we expect, large quakes are less frequent than little ones, but the law **quantifies** this relationship with precision: in any given region and period of time, for every event of magnitude 4, we can expect 10 events of magnitude 3, 100 events of magnitude 2, etc. If the selected region and period of time are large enough, the law has proven to be quite accurate.

In the following pages we will explore this kind of law and the situations where it applies.

Fig. 2 "Magnificent CME Erupts on the Sun—August 31", by NASA Goddard Space Flight Center, http://www.flickr.com/photos/gsfc/7931831962/. License at http://creativecommons.org/licenses/by/2.0/

Many Different Phenomena (Fig. 2)

Another astonishing image is reproduced here: a solar flare that occurred on the 31st of August 2012. Events of this kind are called Coronal Mass Ejections: long plumes of solar material are ejected into space. To better appreciate the image scale, consider that the filament thickness is more or less equal to the earth's whole diameter. Flares impact eventually propagates to the earth, causing wonderful auroras and sometime severe problems with communication satellites.

The physical mechanisms behind solar flares and earthquakes are completely different, but if we collect events and do some statistical analysis, then patterns involved look remarkably similar.

Actually the statistical laws that we are going to analyze are associated to a wide range of phenomena including the distribution of cities' sizes, the frequency and size of earthquakes, solar flares, wars and economic crises, moon craters, people's personal fortune, words in a text corpus, citations received by a publication, movie profits, etc.

In other words there are universal laws that control different phenomena and that are substantially independent of the underlying dynamical details. This observation is called **universality**.

The Bell Curve

In order to appreciate the characteristics of the distribution we are going to explore, it is useful to revise a very well known, ubiquitous and quite different distribution, commonly recognized for its bell-shaped curve: the so-called "*Gaussian curve*", after the mathematician Carl Friedrich Gauss.

To introduce this distribution, let us consider the IQ (or Intelligence Quotient) of a given population. A person's IQ is a score derived from tests specifically designed to assess intelligence. (Actually, the matter is rather controversial and even the bare notion of intelligence is not so well established, but in the next pages we will ignore

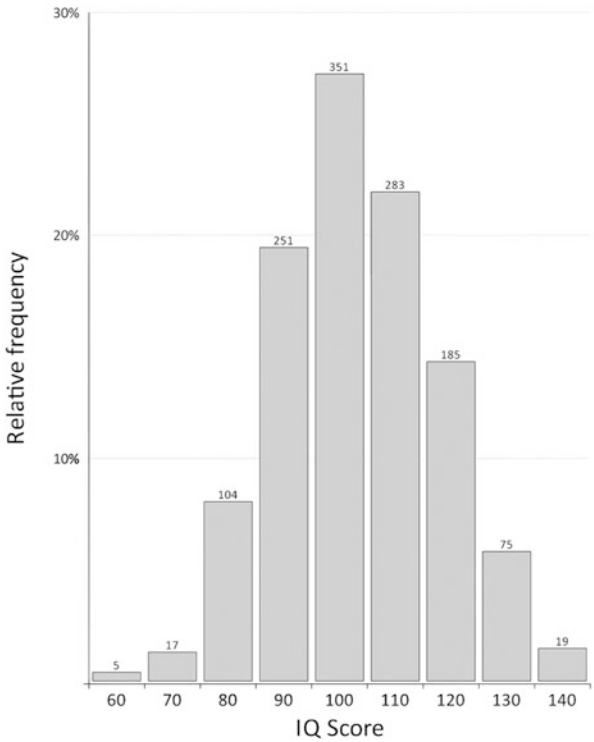

Fig. 3 Distribution of IQ scores of 1,290 boys and girls. Source: Galo dataset, by UCLA Department of Statistics

these concerns. For sure the test is able to distinguish people that are commonly considered very smart from the dumbest ones and, to say the least, it can well measure the ability to solve the test itself: test results are very replicable).

If we plot the distribution of IQ scores of a large and homogenous group of people, we will obtain a histogram similar to the one below (Fig. 3).

On the horizontal scale we can read IQ scores, while the histogram's bins' heights represent the frequency of corresponding scores in the whole population. The graph contains a lot of interesting information. The distribution has a single peak and this peak includes the largest area of the graph: most people's intelligence ranges from 80 to 120 (an IQ of 100 measures by definition the average intelligence) and there are only few very smart people and only few very dumb people. The distribution is rather symmetrical and has no strong bias to the left or the right. We can assume that the IQ 100 mark divides the population evenly in two parts.

(In other words, 50 % of the people are less intelligent than average: this is a trivial consideration from a mathematical point of view, but it is somehow a funny statement, that explains a lot of strange things of the world we live in).

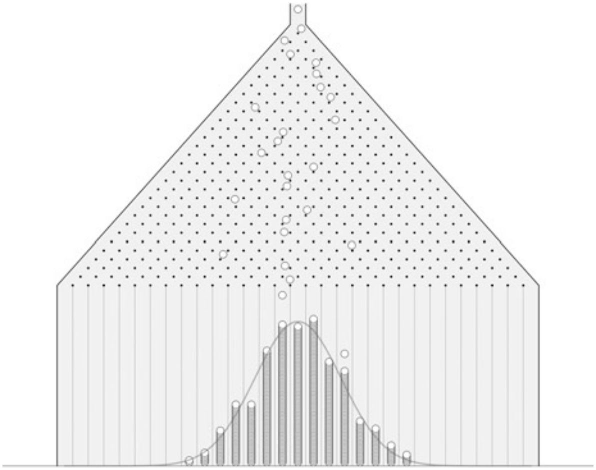

Fig. 4 Galton box (also known as Quincunx or bean machine)

When the sample size is large enough, the actual histogram becomes a very good approximation of the Gaussian curve or, alternatively, the Gaussian curve describes very well the actual data. For instance, using the Gaussian curve, it is possible to compute the expected number of individuals who have an IQ score greater than a given value (e.g. only 25 % people have a score higher than IQ 110).

A huge variety of phenomena follow this kind of distribution. To name a few: people's height or weight, cars speed on a highway, measurement errors, etc. The Gaussian distribution is so common that statisticians call it the *normal* distribution.

It is noteworthy that very different mechanisms, even very complex ones, generate normally distributed data. To understand better the reasons why the Gaussian curve describes well so many phenomena, we can use a very simple model that generates a Gaussian distribution and sheds some light on the subject.

The model can be implemented by building a simple mechanical device or can be easily simulated even on a modest computer. The device is called *bean machine* or *quincunx* or *Galton box* and is made of an inclined plane and an array of pins (Fig. 4). Balls are dropped from a point on the top and bounce left and right on the pins. We assume that when a ball hits a pin, it has an even chance to bounce to the right or to the left. Eventually, balls are collected in a series of bins at the bottom of the plane. The width of each bin can accommodate only one ball.

When a large enough number of balls have been dropped in the machine, the levels of the balls collected in the bins will shape the profile of a Gaussian curve.

It is relatively easy to understand why we can expect a bell-shaped curve. To reach the leftmost bin a ball should bounce left all the times: quite an uncommon event. The same happens for the rightmost bin. On the contrary, there are many paths that reach the central ones. Therefore the central bins will eventually contain many more balls than the leftmost and rightmost bins.

It is possible to demonstrate that this distribution tends to the Gaussian distribution when both the number of balls and the number of bins become very large.

If we consider a much more complex system (e.g. the mind of a person who is taking an IQ test) we can reasonably imagine that the final outcome might depend on the combination of many independent factors just like the final bin position that depends on a sequence of independent bounces on many pins. We may therefore expect the final score to follow the normal distribution.

The Pareto Distribution

The Gaussian distribution is such a good model to describe many real-life phenomena that we tend to assume it implicitly, even when this is not appropriate. For instance we very often refer to the "average value" of some quantity, unconsciously attributing many properties to the average (e.g. we expect most sample units to be close to the average), but for many distributions this assumption is wrong (e.g. a two-peak distribution could have very few observations close to the average). In the following paragraphs we will analyze a number of phenomena that are not well described by a normal distribution.

Let us consider the distribution of wealth. The following graph is derived from the 2010 Survey of Consumer Finances, by the Board of Governors of the Federal Reserve System (Fig. 5). The survey contains data from 6,482 American households. The histogram shows the net worth of the 4,291 households that have a positive value less than 1,000,000$.

The horizontal axis measures wealth. The height of bins measures the number of households having that wealth. This graph is difficult to accept and digest, because it illustrates a tough and selfish society, with many poor people and an enormous spread between the richest and the poorest individuals.

This histogram is very different from the previous one. The most evident difference is the lack of symmetry in the distribution of wealth: there are few rich people but a lot of poor people. Moreover the average wealth is mathematically well defined, but it is not connected with a peak and does not seem to be a very meaningful indicator.

There is another very important difference, although not a very evident one. When comparing the right tails of the two histograms, one can observe that the wealth distribution has a thicker tail, i.e. the frequency of higher wealth decreases, but not as dramatically as in the Gaussian curve (Fig. 6).

According to the Gaussian distribution, raising the IQ from 100 to 110 reduces the number of people scoring more than the given value from 50 % to 25 %. If we make another step of the same size (i.e. we consider only people with an IQ larger than 120), then the percentage drops to 9 %. And only 2 % of the population has an IQ larger than 130.

In the wealth histogram, 50 % of the sample have a worth greater than 189,000$ while 25 % have a worth greater than 996,000$. If we make another linear step of

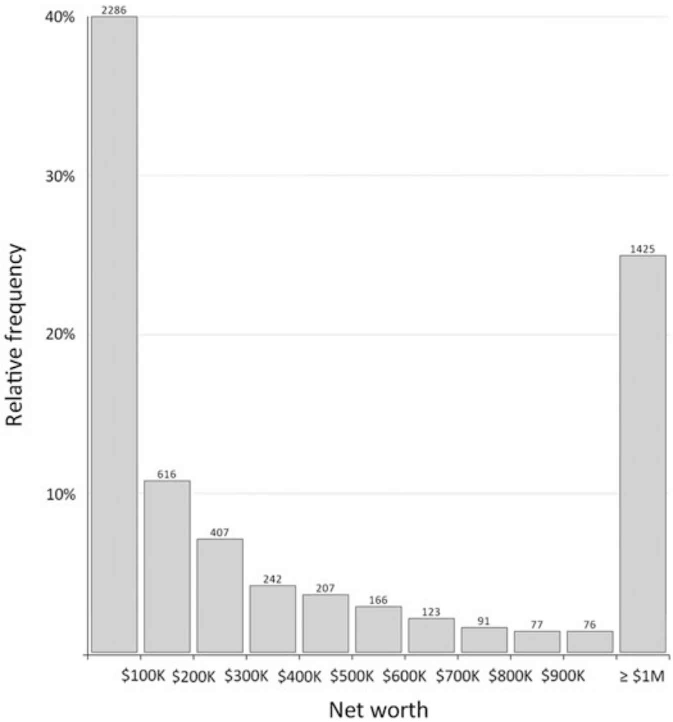

Fig. 5 Distribution of US households net worth. Data source: 2010 Survey of Consumer Finances, Federal Reserve System

807,000$ and raise the threshold to 1,803,000$, we still find found that 20 % of the population are richer than that. And a third step (worth greater than 2,610,000$) reduces the percentage of richer people just to 17 %. Only when we set the threshold to 91'500'000$ the percentage decreases under the 2 %.

The number of people with a 200 IQ (ten "steps" after 100 IQ) is substantially zero, but if we consider the analogous worth of 8'259'000$, we still find more than 10 % of the sample that has at least that worth.

In 1909 the Swiss-Italian economist Vilfredo Pareto proposed to model the wealth distribution with a power law, i.e. a function of the form $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}^{-\boldsymbol{\alpha}}$, where x is the worth and y is the population density for that worth. To avoid the infinite value when $x = 0$ we must consider only values of x greater than a given x_{min} (for the wealth distribution x_{min} can be set as equal to the poverty threshold). The constant α is called the power law exponent and does not depend on the units in which the law is expressed; it must be positive. The A constant is irrelevant: its value is adjusted in order to set the total area to the unit.

This law fits rather well the right tail of the actual data (while the leftmost part of the histogram follows a different law) and indeed has been extensively used in the last century to model wealth and income distribution. Only recently, using a more

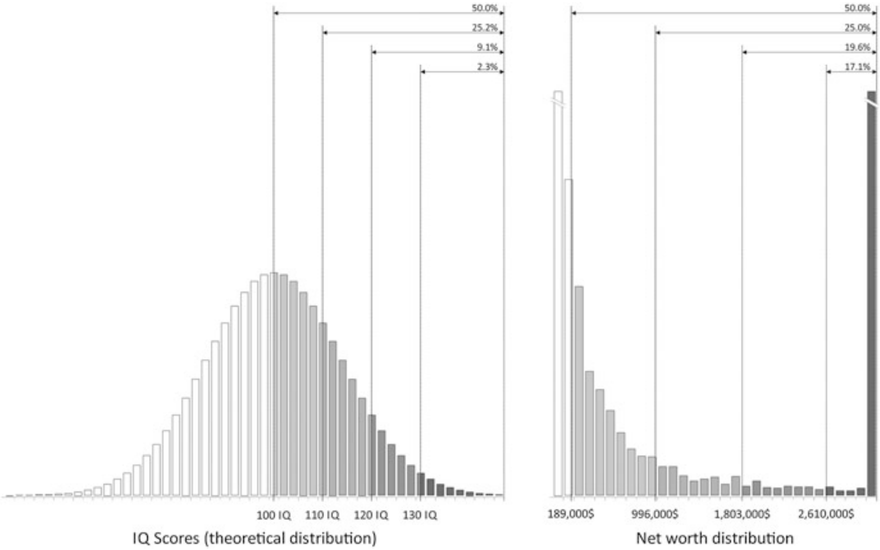

Fig. 6 Comparison between a Gaussian distribution and the net worth distribution

refined methodology, some research shows that a more complex formula must be used to represent more accurately the actual data. Nevertheless the Pareto model is very simple and effective and remains a valuable tool.

The power law has been used to describe all the phenomena listed in the previous chapter and many others. For instance let us consider the text of the *Moby Dick* novel. If we count the number of occurrences of each word in the novel, sort the words according to their frequency and plot rank against frequency, we will get a power law curve with an exponent very close to one. The most frequent word ("the") is roughly twice more frequent than the second one ("of"), and n times more frequent than the n-th word. For larger text corpora the law (in this context it is called Zipf's law, after the American linguist George Kinsley Zipf) is quite accurate.

The particular shape of power law distributions can have a tangible economic impact and can be used to leverage profits. As we have seen, a significant share of population lies on the tail. Popularity graphs of some goods (e.g. a list of best-selling books) typically follow a power law. It can then be profitable to sell small volumes of many items instead of concentrating only on top sellers. These 'rare' items are individually less popular but still meet many customers on the right tail, thus bringing substantial total profit to the publisher.

Power law graphs are typically plotted on a double logarithmic scale (Fig. 7). On this scale power law curves become straight lines, making it is easy to see how close actual data are to the model. Moreover the power law exponent can be quickly estimated because it becomes the slope of the straight line.

One of the most remarkable properties of power laws is their self-similarity or their lack of a typical scale. The Gaussian curve peak is a significant landmark that

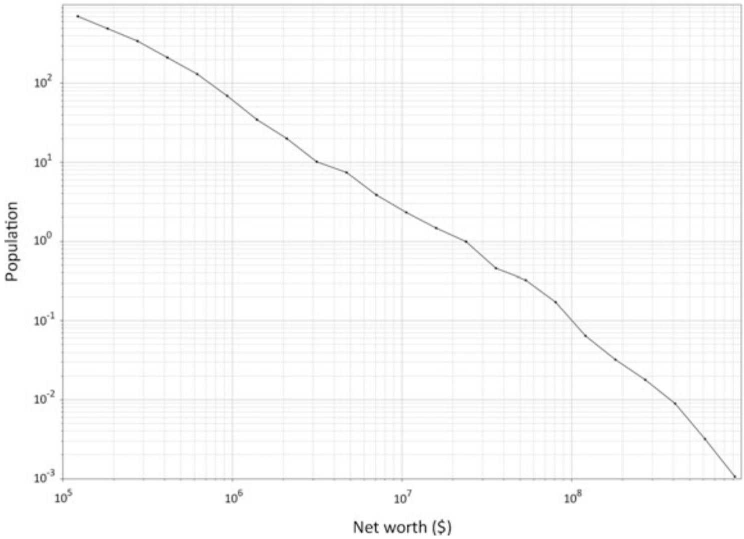

Fig. 7 Net worth distribution on double logarithmic scale. (Logarithmic binning is used to reduce noise)

defines a natural unit. Furthermore the width of the peak depends on the actual unit used in the graph: a unit change makes the peak wider or smaller. In power law graphs there is nothing similar. Indeed, if we change the scale of the argument, that will result in rescaling the whole function. In other words, if we zoom the curve on a section we find again exactly the same shape. Remember the wealth distribution: we had already noticed the large spread between the rich and the poor, but it is remarkable that also in the richest group there is a huge spread. In the list of world billionaires compiled annually by the Forbes Magazine, the first person in the list is 73 times richer than the last one. And if we examine different parts of the list we find the same shape: the 100th person in the list is three times richer than the 400th and the 200th person is three times richer than the 800th.

Another typical feature of power laws is also interesting: richest people are a small fraction of the whole population, but they are so wealthy that they own a significant share of total wealth. For instance, in the billionaires list, the richest 35 % own the 70 % of the total wealth in the group.

This rule, called the "Pareto Principle", is a very general one. Numbers are metaphoric and not rigidly determined. The overall idea is that a large part of the total value is controlled by a small group of units or individuals. Indeed the principle is also known as the *80–20 rule*.

In 1906 Pareto observed that in Italy, 20 % of the population owned 80 % of the country. A century later the situation has not improved. According to a survey published in 2008 by the United Nations University, the world distribution of

household wealth is even more inequitable: the richest 2 % of the world's population controls more than half of global wealth.

Pareto later realized that the principle's field of application is much larger: for instance he noticed that 20 % of the pea pods in his garden contained 80 % of the peas. The business manager consultant Joseph M. Juran applied the principle to quality issues (e.g. 80 % of the problems are caused by 20 % of causes) and named this principle after Pareto. Juran also called the principle "vital few and trivial many": in any matter, a few are vital and many are trivial. The principle works in both ways: in a team 20 % of the people will cause 80 % of the problems, but another 20 % will be responsible of 80 % of progress.

Sandpile (Fig. 8)

To understand the universality of the Pareto distribution, we can examine some simple models describing different phenomena that exploit a power law. One of the simplest and well studied is the *sandpile model*. In some sense it is similar to the *bean machine* that we used to explain the Gaussian distribution. In the following paragraph we will implement it with a computer simulation.

In the model some grains are dropped on a circular platform of a given radius. For the sake of simplicity we assume that grains are cubic and that their positions are aligned with a square grid. When a grain lands on another grain they form a tower. When a tower reaches a given height in comparison to its neighbors its top four grains roll down and land in the four adjacent cells of the square grid, possibly on top of other, smaller towers. As a consequence one or more of these towers can grow beyond the threshold, causing other grains to fall. Eventually a pile of grains (made

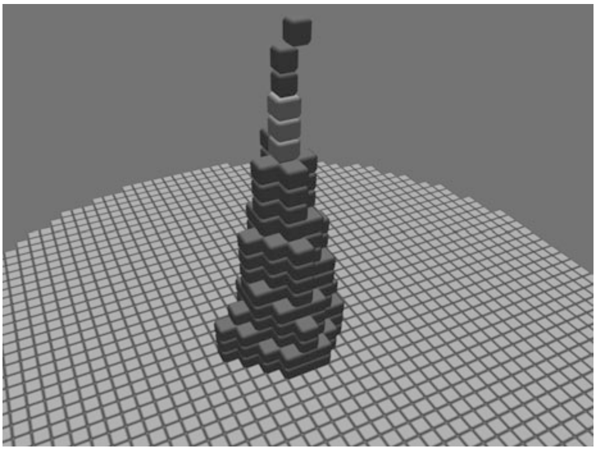

Fig. 8 A sandpile computer model. Grains are represented by cubic boxes

Fig. 9 Scree slopes, the Vestrahorn, South Iceland; "Islan Vestrahorn 01" by Zairon, http://commons.wikimedia.org/wiki/File:Island_Vestrahorn_01.JPG. License at http://creativecommons.org/licenses/by-sa/3.0/deed.en. Desaturated from original

of many adjacent towers) will grow (Fig. 8). When an additional grain is dropped on top of the pile it can just land and remain still, or it can trigger an avalanche involving many grains.

When the pile is large enough, grains in excess drop off the platform and get lost. At that point the number of grains in the pile reaches a sort of dynamic equilibrium, oscillating around a given value. The pile can grow, forming steep walls, which eventually collapse forming avalanches that reduce the pile. On the other side, each avalanche tends to reduce the inclination of the hill sides and therefore favor the pile growth.

These two opposite phenomena adjust the slope to a very precise value, all around the pile, in spite of the cubic shape of the single grains. The pile tends to assume a conic shape, with a sharp top (very different from the round top in the bean machine). Indeed real sandpiles do have a conic shape. Also screes at the base of mountain cliffs follow the same pattern: their slope is almost perfectly constant all the way long (Fig. 9).

The slope value eventually reached by the sandpile (or by the screes) is *critical*. Avalanches are small on slopes under this value. On the contrary, when the slope is critical all avalanches sizes are possible. In the simulation a single grain can trigger all possible consequences, from moving no other grains to causing a catastrophic avalanche that involves a large part of the pile.

The histogram of avalanches frequencies against size follows reasonably well a power law (Fig. 10).

It is possible to create an even simpler model with the same characteristics, but more abstract and suitable to mathematical analysis. Let us consider a cellular automaton working as follows: on a bidimensional grid each cell is associated to an

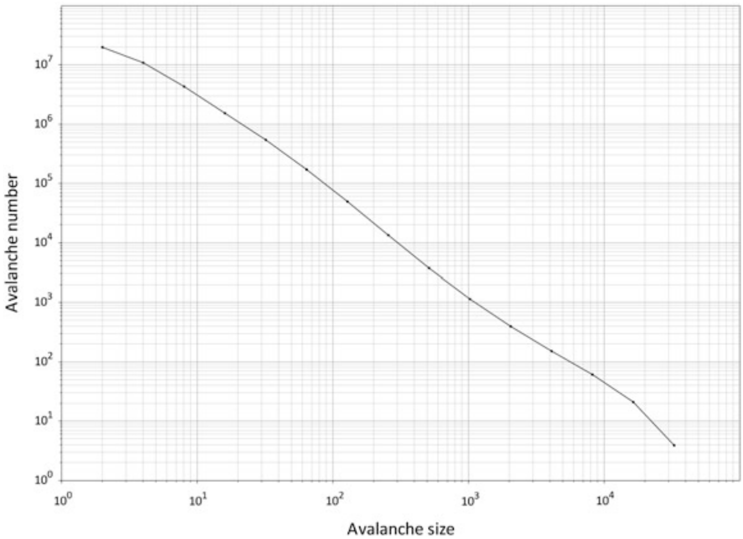

Fig. 10 Distribution of sandpile avalanches on double logarithmic scale. (Logarithmic binning is used to reduce noise)

integer value in the 0–3 range. This value represents the cell's instability (it roughly corresponds to the height of the grain towers). When a cell reaches the value of four it collapses: its value drops to zero and values of the four adjacent cells are incremented by one. Of course this event can trigger an "avalanche" if some of the adjacent cells already have a value of 3.

The distribution of avalanches (frequency against size) follows a power law.

It is possible to demonstrate that the order in which cells are processed does not change the final result. This is very convenient to speed up the simulation and simplify analytical computation.

The following pictures show the grid after four millions iterations (Fig. 11). The bottom image is a close up revealing interesting self-similar structures.

Forest Fire

It is possible to come across many features of the sandpile model (the power law, the avalanches and the critical state) when modeling a completely different phenomenon: forest fires.

Two-dimensional cellular automata are convenient here as well. We represent the forest with a square grid of cells. A cell can be empty or contain a tree. New trees can appear in empty cells. Sometimes lightening can strike a tree and start a fire. Fire spreads across adjacent cells: if a burning tree is close to another tree, then the second tree starts burning. Extinguished fire leaves empty cells (where new trees

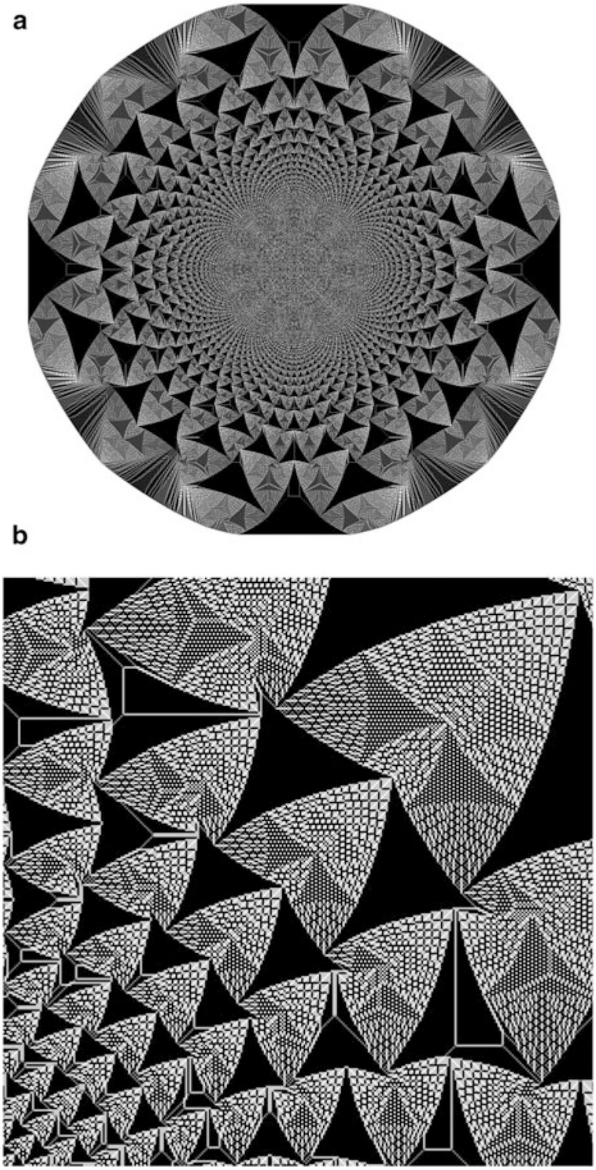

Fig. 11 (**a**) Sandpile after five million grains have been dropped (**b**) close-up

can grow). The entire system is defined by two probability values that control tree growth and fires. We can tune up these values: we want the forest to grow very slowly (with respect to the speed of fire) and lightening to be rare.

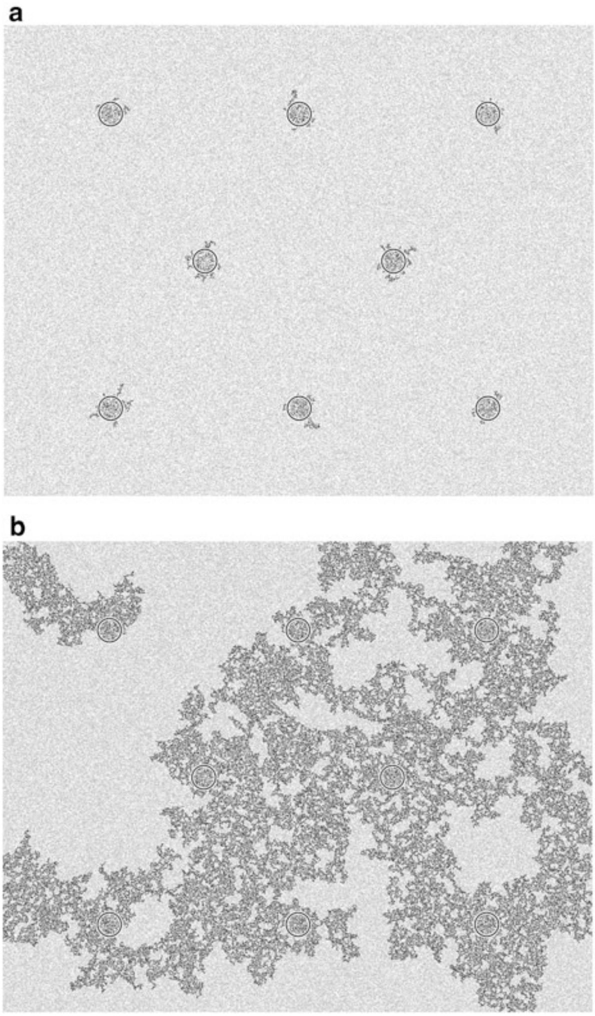

Fig. 12 A cellular automata model for forest fire. *Gray cells* are trees. *Black cells* are burnt trees. Fire is started at the circles. (**a**) tree density under the critical value: fires tend to extinguish (**b**) tree density slightly beyond the critical value: fires can spread to the whole forest

At any time, trees are organized in clusters of connected cells. When a fire starts at a given tree, all the trees belonging to the same cluster will eventually be destroyed. Therefore the expected fire size is equal to the expected cluster size.

If tree density is low, then the average cluster size is low: most clusters have one of few trees.

In that condition fires extinguish immediately and few trees are damaged. Beyond a critical tree density value, large clusters, spanning the whole forest, begin to appear (Fig. 12).

As in the sandpile case, the system self-organizes in a critical state: at low densities, fires damage few trees and the forest can grow and became denser. At very high density, fires propagate across large parts of the forest and are very effective in lowering tree density.

The dynamic equilibrium of the forest is well understood by forest keepers that avoid extinguishing all fires, because this behavior would increase tree density well beyond the critical state, with the risk of a catastrophic fire, which could almost destroy the whole forest.

When density is close to the critical state, all fire sizes are possible. Again if we make a frequency/size histogram, we will recognize a power law.

Network

Next example is another illustration of power law universality: again a phenomenon completely different from the previous ones, but with similar statistical features. Moreover in this case we will be able to move from a Gaussian-like distribution to a Pareto-like, getting more insight of the basic mechanisms behind power laws. We are going to consider a human generated phenomenon: the Internet.

Today the network counts more than 1,000 million of interconnected nodes and is used by more than 2.4 billion people. Internet began in the 1960s as a research project funded by the Defense Advanced Research Projects Agency (DARPA). In the 1980s it started evolving well beyond its primary research roots and the number of hosts grew dramatically, breaking 100 nodes in 1977, 1,000 in 1984 and 10,000 in 1987.

The invention of the World Wide Web in the early 1990s caused an explosion in both business and personal use. During this rapid growth period, the network's topology deeply transformed. The first topology was very regular; each node had more or less the same degree, i.e. the same number of connections and therefore the same importance. For this reason the network was very robust against single nodes' failures. On the other side the network was relatively wide and the average distance between two nodes was proportional to the total number of nodes.

Today, Internet offers a completely different landscape. Some nodes have a very large number of connections while the vast majority has very few. Moreover the average distance between two nodes (the "degrees of separations") is surprisingly low. In 1999, according to a study, more than 800 millions web pages could be connected by no more than 19 clicks. The network is a so-called *small world* network. The presence of few *hubs*, i.e. nodes with very many links, allows distances shortening.

If we draw a histogram showing the number of nodes with a certain degree (i.e. the number of connections) we get a power law. The technical architecture (the TCP/IP protocol) does not enforce this rule and the Internet growth has not been

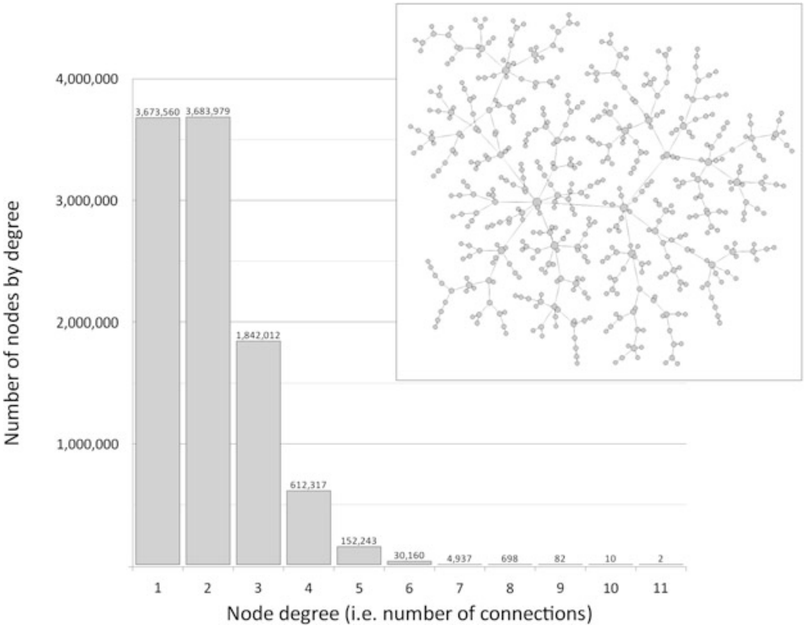

Fig. 13 Degree distribution histogram. The network has ten million nodes, each one randomly connected with another one. The top-right image represents a smaller network with the same structure

regulated by a central authority. So it is interesting to explore the reason why the network spontaneously evolved this way.

To investigate the pattern, we can create a simple computer simulation that builds a random network. The program creates a given number of nodes and then links each one to another node, selected at random. Nodes in this model will have an average number of two links and the number of nodes with many links will remain very low. In a simulation with ten million nodes the maximum degree is 11 and the distribution of nodes by number of links looks like a Gaussian right tail. (More precisely, it is a Poisson distribution). In the above simulation the average distance between two nodes is more than 2,500 steps (Fig. 13).

We can introduce a small change to the algorithm that will cause a large difference in the resulting distribution. As in the previous method we link each node to another one, but we change the way in which we select the other node. Instead of randomly selecting a node with equal probability, we randomly select an existing link and then we choose (always at random) one of the two linked nodes.

This procedure chooses more frequently those nodes that have a higher degree (i.e. with a larger number of connections). This strategy has been called “preferential attachment” or “rich become richer” and reflects a common real-life behavior: it is

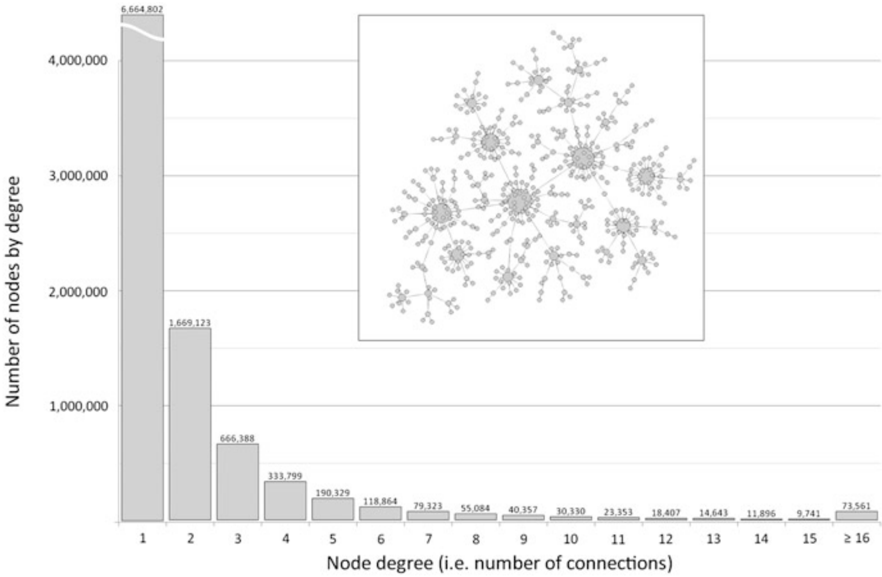

Fig. 14 A slightly different random network that exhibits a power-law distribution

more probable that a newcomer connects with a node with many connections (e.g. google.com or wikipedia.org) than with less known nodes.

When we examine the generated network we find many differences with the previous one. The average degree is still two, but there are quite a number of nodes with very many connections: in a ten-million-nodes simulation the maximum degree is larger than 7,000. Moreover the average distance between two nodes is definitely lower: less than 17 steps (Fig. 14).

Finally, if we plot on a graph the node degrees by number of nodes on a double logarithmic scale we notice that a large part of the graph is well approximated by a straight line: the distribution follows a power law.

It is interesting to note that the "rich become richer" mechanism is clearly connected to the wealth distribution, but also to the other examples. For instance in the forest fire model a large cluster of connected trees has a large boundary (i.e. many empty cells touch some trees of the cluster). So when a new tree appears, it is more probable that it will become part of a larger cluster instead of a smaller one. As a consequence, large clusters become larger and larger and one can observe huge differences in cluster sizes.

A key element in determining a power-law distribution instead of a Gaussian distribution is the events interdependence: the number of connections affects the probability to receive a new connection. To the contrary, we have seen that the processes described by the Gaussian distribution are made of independent events.

Slider Block Model

To conclude this survey of various systems that exploit a power law we come back to the beginning of the article and face the challenge of simulating earthquakes.

Although they are the most complex phenomenon among those listed above, some simple models can be used to investigate the related general laws, as the Gutenberg-Richter law that we have mentioned in the first paragraph and that correlates earthquakes frequency and intensity.

One of the simplest earthquake models that can simulate the Gutenberg-Richter law has been proposed by Huang and Tucotte in 1990 and it is called the *spring slider block model*.

The fault is modeled by two parallel plates that move against each other with a constant speed. We can assume that one plate is still, while the other is moving. A number of blocks connected with springs represent rocks in the fault. Other springs connect the blocks to the moving plate. Blocks can slide with respect to the still plane, and they are subject to a frictional force. The frictional force is higher when blocks are stuck to the plate and lower when they slide. Because of this force, blocks alternate between two states: *stick* and *slip*. In the stick state, blocks remain still until the elastic force overcomes the static frictional force; at that point, blocks start sliding, pulled by springs. When the elastic force decreases below the dynamic frictional force blocks stop and the cycle repeats. An earthquake is represented by blocks' movement in the slip states.

Even the very simple system with only two blocks is interesting. For some parameters range, with different frictional forces for the two blocks, behavior is chaotic (the slip/stick steps repeat with different time intervals, without simple periodicity).

When we simulate a system with many blocks, the slip of one block can suddenly increase the elastic force applied to its neighbors, possibly causing them to slip. These slips can in turn induce other slips; in other words this system is able to generate "avalanches" (Fig. 15). A single slip event can have no consequences of can eventually provoke a movement of many blocks. The number of moved blocks can be used to determine the "earthquake" magnitude. If we plot the earthquakes frequency by their magnitude, we will again discover a power law.

Conclusion

We have considered a number of different phenomena involving interactions of many agents. Statistical features of these phenomena cannot be described by a Gaussian curve and power laws offer a much better approximation. Some noteworthy characteristics of these system are thick tails (i.e. the significant amount of event well far from the average), the presence of events of all sizes and the lack of a typical scale, events interdependence (events can cause other events), the distribution's

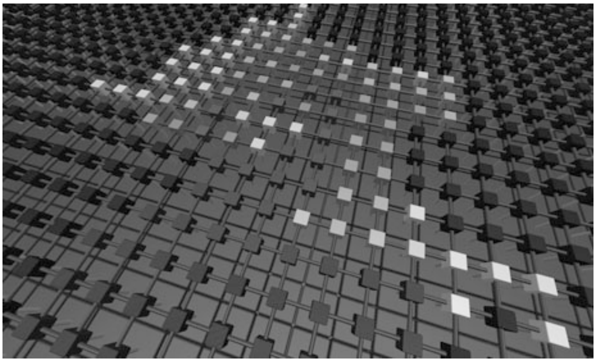

Fig. 15 A large earthquake strikes in the computer simulation. More than 60 blocks are involved and slide at the same time

self-similarity and the appearance of the 80–20 rule (80 % of effects are due to 20 % of causes).

In the last century, mass production and the related industry and economy favored the standardization of products, consumers and workers, shaping a world well described by the Gaussian curve.

As to mark the change, the new century started with a pair of severe economic crises and crises are phenomena well described by a power law.

Today the availability of fast and cheap communication, globalization, and the evolution of niche markets create a landscape where distributions' tails are important and the 80–20 rule is self-evident almost everywhere. It is possible that, to better understand this century, the Pareto distribution will be no less useful than the Gaussian curve.

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Part VI
Mathematics and Cinema

Representation of Space-Time in the Cinema: The Art of Editing

Edouard Blondeau

"The cinema is the truth 24 times per second"
Jean-Luc Godard

The cinema is inherently linked to a notion of space and time. Cameras, even digital, work on a 24 frames per second; scenarios describe for each scene when and where the action takes place . . . The editing is one of the tools who permit to represent the impression we have of space-time.

The Birth of Editing

At the end of the nineteenth century, the first films of Lumière brothers and Edison are all short filmed scenes. They last less than a minute and are always presented in the same layout: a roll of 35 mm film about 20 m (65 ft), which is impressed on a single shooting comprising a single plan.

In 1896 a shooting incident would have given a new idea to Georges Méliès. While filming a busy street, the crank of his camera would have been blocked and restarted a few moments later. When developing Méliès would have found that a bus had suddenly turned into hearse. Méliès therefore decides to use this method to tell fantastic short stories. Méliès turns in 1896 a first film according to this principle entitling *Escamotage d'une dame au théâtre Robert Houdin*.

It is only the beginnings of editing, exercise currently called "shot-edited". In 1900, George Albert Smith shows the public the first film, *Grandma's Reading Glass*, containing several successive shots telling the same story in chronological order. The film editing is born. But Smith does not consider having invented or discovered anything but is only inspired by the narrative of magic lanterns.

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The Fast Forward

The first example of fast forward dates from 1898 in a film directed by Georges Méliès and called *Carrefour de l'Opera*. Unfortunately the film was lost and we don't know why Méliès accelerated the images, although his repertoire suggests a comic or fantastic goal.

However we feel that all the early silent films used this principle to give a comic effect. These films shot at 16 frames per second were also screened at this rate. The judder observed when passing at 24 or 25 frames per second is not the effect of accelerated described here. This jerk is due to lack of images when switching to 24 or 25 frames per second. Some images are necessarily doubled for the transfer. The total duration of the film remains almost unchanged.

Since fast forward is still used for various purposes. Firstly the most obvious is to accelerate a movement or an event that is extremely slow. Lots of documentaries use this purpose to see a flower bloom or glaciers melt.

Besides the documentaries, fast forward is used in a narrative purpose. Thus with a burlesque music, Benny Hill very regularly use accelerated images to his televisions shows chases. The comic effect is pronounced.

Cédric Klapisch uses fast forward in a different way in *L'auberge espagnole*. When Xavier arrives at the Finance Ministry, the images when he walks through the building are accelerated. This highlights the vastness of the building, but the jerk of images also gives the impression of a place where the work is repetitive and few fulfilling.

In *Vanilla Sky* Cameron Crowe uses the same process with a short-cut editing. At the beginning of the film when David Aames lives in a nightmare of being alone in New York, the anguish of the character is accentuated by fast forward.

Fast forward also helps to isolate a character of its environment. In Danny Boyle's *Trainspotting*, after his withdrawal from heroin, Renton plays Lotto. Everybody around him moves and speaks very quickly while he remains stonily indifferent. The concept of time is completely different for him and the other characters: Renton bored while everyone has fun.

A surprising case of fast forward is the French trailer of *Femme fatale* by Brian de Palma, edited by Sonia Tout Court agency. The film is fully accelerated to last 2 min. Only a few shots with the main characters are at normal speed. This process accentuates the tension and by discovering this trailer, Brian Palma himself congratulated the authors. A few years later, Jean-Luc Godard also uses this method for editing the trailer of his film *Film socialisme*.

The Slow Motion

Slow motion is invented in 1904 by August Musger. Used to show very rapid action, it is used at first to glorify the heroism and effort. Hugh Hudson uses it regularly throughout of *Chariots of fire*. Especially during the final race when Lidell is at a few meters from the finish line.

Slow motion is widely used in action movies in order to punctuate the scenes. John Woo uses it regularly in his films, often coupled with shots of doves flying away—including *Mission: Impossible II* when Ethan Hunt enters the bunker to face dozens of enemies.

It also allows to enhancing the drama of production. Thus, in *Platoon*, Oliver Stone uses this effect to accentuate the desperate race of Sergeant Elias pursued by the enemy.

The climax of slow motion is the bullet time which is obtained with using a series of cameras placed around the action. They are triggered simultaneously, or with a very short time differential, allowing after editing to give the illusion of a camera moving freely around a slowed to the extreme or fixed action. The technique used for the first time in the short film of Emmanuel Carlier, *Temps mort autour de Caro & Jeunet,* is popularized by *Matrix* from Andy and Larry Wachowski. When Neo and Trinity are fighting against the matrix, we have the impression that their movements exceed the speed of light.

This effect, while praising the effort, courage and enhancing the drama of situations is obviously used in comedies and parodies. In *The Mask* of Chuck Russell, the scene where the mask flies across the room and no one can catch it except the dog, is in slow motion thus increasing the funny side.

The Flashback

This method is popularized by Orson Welles *in Citizen Kane*. The entire plot is based on a series of flashbacks that make it possible to better understand the personality of Charles Forster Kane.

In *Memento*, Christopher Nolan uses the flashback to the extreme by telling the story of Leonard upside. The principle works because Leonard, the main character, has a memory of 15 min and he wants to know why he is in this situation.

The Flashforward

Flashforward is the reverse process of flashback. Often used at the beginning of movies, this effect presents either a future which could be changed or focusing the audience on how it happened. For example *Fight Club* from David Fincher begins with the start of the last stage when the narrator is attached on a chair in the last floor of a building.

Use of the most successful flashforwars is made in *La jetée* directed by Chris Marker whose *Twelve Monkeys* from Terry Gillian is the remake. The hero remembers the murder of a man he saw child. This man is only himself years later.

The Split Screen

Split screen is a visible division of the screen in several simultaneous shots. For the first time used in 1913 in *Suspens* from Lois Weber, the split screen is popularized at the end of the 1960s by several studio-made films.

Brian de Palma is one of directors who use regularly split screen. In *Blow out*, the scene showing Jack understanding what happened is split. One shot shows Jack listening to the sound of an accident and the other shot shows the accident mentally reconstructed by Jack.

The Jump Cut

This type of edit gives the effect of jumping forwards in time. It is a manipulation of temporal space using the duration of a single shot, and fracturing the duration to move the audience ahead.

Jean-Luc Godard is the first to use it in a narrative way in *A bout de souffle*. When Michel speaks in the car, several jump cuts give the impression that the journey lasts because we don't know how many times pasts during the ellipsis.

Extension of Time

The editing permits to make the story last longer than the reality. For example in *Goldfinger* from Guy Hamilton, the bomb explodes in 4 min. But these 4 min of the countdown lasts in reality 8 min which is the time necessary for James Bond to fight Oddjob and for the army to penetrate inside Fort Knox and neutralize the bomb.

Time to Conclude

So the editing permits to play with the notion of space-time. However other tools like set, sound or special and visual effects are also used to divert space-time. In *THX 1138* from George Lucas the set of jail is uniformly white so that the main character loses his benchmark for judging the distance or the flow of time. The visual effects of *2001: a space odyssey* from Stanley Kubrick give also the impression to assist to the big bang. And in *La haine* de Mathieu Kassovitz the beeps of the clock sound like a countdown.

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Part VII
Mathematics and Theater

Searching for a Digital Performing Art

Claire Bardainne and Adrien Mondot

Introduction. Who Are We?

Ever since its foundation in 2004, the company has been involved in digital arts and performing arts, focusing on the sensual relationships between moving bodies and moving digital objects.

The company creates shows that mix the imaginary with the real and virtual. Performance is the main concern which subtends all the company's creations, whose specific designs rely on, and stem from the definition of their own customized digital tools which are part of the creative process. The company puts man and body at the heart of the technological and artistic investigation, and searches for a digital performing: moving, organic, ephemeral, sensual.

Adrien Mondot and Claire Bardainne restarted the company in 2011 and today they carry on the company work together, jointly designing creations that range from performances to installations, on scales both small and large.

Adrien is a computer scientist and a juggler. He investigates movement at the meeting point between the art of juggling and digital innovation. Inspired by the movements of balls, he designed digital tools that make it possible to generate sensual interactivity.

Claire is a visual artist, graphic designer and stage designer. She works on the form and its meaning. Her research focuses on the imagination of images and graphic signs, especially when those create space.

Here are the main principles from which the artistic project stems:

- A sensual digital art subtending performing arts. In order to preserve the power of the actor's presence on stage, while inserting it into the digital medium, all images are generated, calculated and aired live.

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- The digital writing and the reading of the digital sheets. Instant sensual perception, although sticking to a precise story, is paramount in the realm of a musical approach to images and software.
- Techniques of interactivity going beyond technical challenge. The use of capture devices from the world of video games (for their calculation speed and reactivity power) combined with virtual puppetry (for its fantasy and human improvisation vitality) makes a sensual experience possible for the audience.
- Reference to the experience of unconscious movement. The digital modelling is inspired by the observation of nature. An intimate, unconscious experience of movement suggests an imaginary space and transforms abstract graphic signs into reminiscent worlds.
- Organizing coincidences. Gesture enters into the image in a relationship of reasonableness and consistency. Building strong connections between digital objects and real objects makes it possible to transform the virtual into an extension of the real.

eMotion: The Software Basics

eMotion is the main software used for the performances. It was designed by Adrien Mondot, who has developed it progressively since 2005, through a series of tentative experiments always related to artistic research.

Running on OS X, eMotion is an application meant to **design interactivity between objects and information**. It uses physical models to build animated situations.

If we think of the art of juggling as the controlled arrangement of the movement of objects in space and in time, then we might say that this software makes digital juggling possible, while still preserving the same intuitive quality as the so-called "real" juggling. This software suggests applications of the movement and possibilities of juggling to much more than balls, but with the same tactile feeling.

What Is an Object?

For eMotion, an object is something which has a mass, and is subject to some physical laws freely inspired from reality, but not necessary realistic. The physical laws of object motion are simplified and modelled through mathematical formulas that computers can handle, then translated into explicit parameters such as vibration and air friction.

It is therefore possible to apply forces to objects, locally or globally: gravity, friction, oscillation, elasticity . . . the sum of the applied forces will determine the

movement of the objects. In fact the user is invited to define the causes of movements more than the shapes of movements (unlike in the *key frame* animation system).

Combining all these parameters together makes it possible to create a wide array of movement features. The number of parameters is finite, but the number of possible combinations is infinite. Thanks to our memory of sensual experiences and our intimate familiarity with object motion, we create, using movement imaginary.

We therefore have little control on the exact shape of movement since we deal with global forces, and thus we have a macroscopic control of behaviour.

Physical properties of an object are not related to its graphic aspect: a big object will *a priori* move just like a small one, a dot like a letter. Basically, we are dealing with the physical model of the dot, with simplified physics, subjectively hybridizing the mathematical and physical worlds.

An object belongs to a setting.

If the basic object is the dot, any animation can be applied to any graphic item: photos, graphic objects, letters, and so forth.

What Is an Information?

Information is something that can be transformed into a series of numbers. For instance:

- the x and y coordinates of the mouse: bi-dimensional information;
- the flow variation of a thermometer: mono-dimensional information;
- an oriented object in space: six dimensions (3 for the position and 3 for the rotation);
- sound: mono-dimensional information;
- a chain of characters: each letter is represented by a number in the ASCII code;

What Is Interactivity?

Designing interactivity means creating relationships, ranking, linking, connecting certain parameters of the physical model to pieces of information. For instance, the mouse pointer attracts particles, thus we can link friction to temperature: the hotter the quicker. But we want to find alternatives to the use of the mouse.

Animations are done live, in real time, and they can be linked to signals coming from the real world. Information is instantly transferred to the movement of virtual objects. In order to do so we need:

- electronic captors, sound or tactile captors, such as iPods, infrared pens or tools used in videogames such as the Wii remote, Kinect for Xbox;
- body motion captors, where with a pen or with the fingers one can create solids, voids, can assemble, scatter, drop or bounce back. To some extent it is

as if images were animated like puppets, in order to make them move with the dancers, thus providing an illusion of interactivity.

What Is Writing?

Writing entails completing scripts, conceiving developments, defining relationships and organizing them in sequence. It involves delineating the timeline of all the relationships: scripts fix the physical states of affairs and their relationships with information. The progression of the show is due to the switching from state to state, which creates real choreographies of virtual objects.

Applications: Shows and Installations

All these possibilities converge towards the same goal: to build shows for an audience.

Here are two examples:

The danced performance *Hakanaï*/Created in 2013.

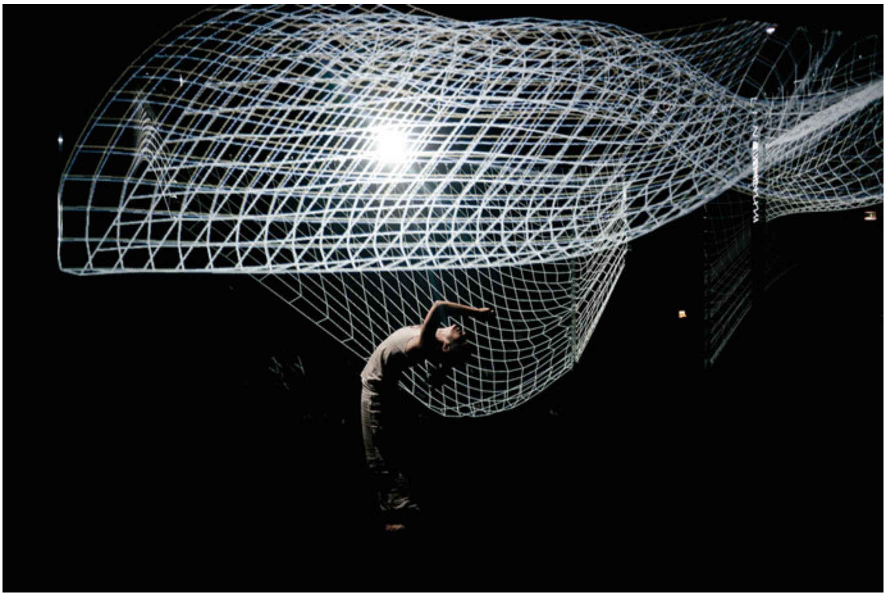

Photo from the show *Hakanaï* ©Romain Etienne

Hakanaï is a choreographic performance for a female dancer moving in a cube of images animated in real time by a digital player.

Hakanaï: fragile, evanescent, temporary, between dream and reality.

In Japanese *Hakanaï* qualifies what is transient and does not last; A very ancient word, it suggests an intangible substance related to the human condition and its frailness but also to nature. Its ideogram combines two elements, one meaning 'man' and the other meaning 'dream'. This peculiar symbolic and poetic collage is the starting point of this sheet music for a dancer and moving images.

A cube of white tulle fabric defines the space of this performance. Four synchronized videos project on the fabric an ever changing graphic reality, generated in real time and played by an artist as if it were a digital sheet music. The body of the dancer reacts to these moving images, simple abstract black and white shapes. The sound is also live, thus creating a total synaesthesia of sound-image-dance.

The installation is meant to be performed in spaces that are not necessarily regular theatres, and where the relationship stage/house is not the usual one. The audience discovers the show step by step. They first discover the installation by walking around it and entering the world of projected images. Then, at the dancer's arrival they take their places all around the structure to view the performance. At the end of the dance, they may once again spend time wandering inside the installation, in the very place where the dancer has just performed.[1]

The exhibition *XYZT, Les paysages abstraits*/created in 2011.

This exhibition-promenade comprises some ten interactive installations. It is a promenade, intended as a stroll through another nature, whose form is halfway between geometric and organic. Fragments of real sensations, such as walking in the grass or touching sand with the hands, are transferred into a digital world made of lines, dots and letters. The stroll goes through modules of various scales: some are large and panoramic, while others are smaller and more intimate; all offer a variety of sensations. In a continuous flow, the movements of forms and signs respond to each other in a graphic, kinetic, minimalist and dreamlike language. All of them are based on models of physical behaviour and therefore lend their suggestive power to reality. The installations either inquire innovative interactivity or rely on ancient illusionist techniques, they play with the dimensions of space, they seek new possibilities from combinatory and generative works. Overall, however, they offer the spectators the possibility to live and experience a space through deliberately basic feelings, simple manipulations which allow them to build forms, draw the world, borrowing from the art of juggling the pleasure of a game free of rules, and from science the fervour for curiosity and discovery. The relationship formed with this other nature invites one to become part of it, instead of being its creator; anyone can then pretend to be the protagonist in an innovative comprehensive vitality.

In this series of installations, a sensual artistic experience involving the body of the spectator is offered to the audience. Walking through the space at their own chosen speed, the spectators move in a world which interacts with them, they

[1] Trailer: http://www.vimeo.com/amcb/hakanai

Photo from the exhibition *XYZT, Les paysages abstraits* ©Laurence Fragnol

encounter a substance that is virtual, digital, vibrant, ephemeral, moving, printed in time and in space, generating gestures and sound, expressive of some sensibility towards the world.[2]

We are definitely trying to build a digital performing art, and thus:

- we organize coincidences between the body and moving images, physical gestures and digital movements. More broadly, we try to create synaesthesia where the many pieces of information sent to our senses (graphics, light, sound) come together;
- We rely on live performance, and handle images with music. There are no pre-registered videos, but there are real time performances of sheet music.
- We refer to recollection, experience and unconscious memories of natural movement and thus make possible the emergence of poetic feelings.

[2] trailer: http://http://vimeo.com/amcb/xyzt

Conclusion: Writing with Images

The language that we are trying to define is based on relationships between video images, body, space and sound.

The digital realm has disrupted contemporary creation in performing arts, and thus we feel like explorers of new worlds, and we call the mark we make in this new territory ‘writing’. In our work, as in land art, we write with the elements, we play with nature, with its forces and tensions, although here fire, snow, sand, water, smoke and pebbles are digital, virtual, and endowed with a vitality of some other kind. We must therefore deal with the forces of this digital nature and control its energy. With the projected images we work with rhythm, with the intentions which lie at the root of movement. We write with the love of movement.

At the beginning of the twentieth century, when architecture started to make use of glass, steel and concrete, architects first used these new materials in the shape of the same old forms. Then, slowly, the structural characteristics and new shapes that were inherent in the materials started to blossom forth and transformed architecture and living.

Today we are trying to understand the essence of the writing composed in digital media, which are still young but yet are a large part of our everyday lives and artistic creations. If we think of the digital image as a tool for story telling, then we might say that we are trying to understand what its specific lexicon is. Perhaps we will realize that this medium may not always be just the tool, but also the message. Fire may be a tool for cooking food, or forge a sword, but it can also be contemplated and can trigger imagination. Rather than using traditional narrative techniques or inserting pre-existing contents, we try to sense what paradigm is brought by this medium. Our research is slowly leading us to the realization of how much the moving image is space, image is reality, and the body is part of it. Writing with a live digital image makes it possible to inspire emotions and sensual experience, and paves the way to a new approach to performing arts.

Translated from the French by Sylvie Duvernoy

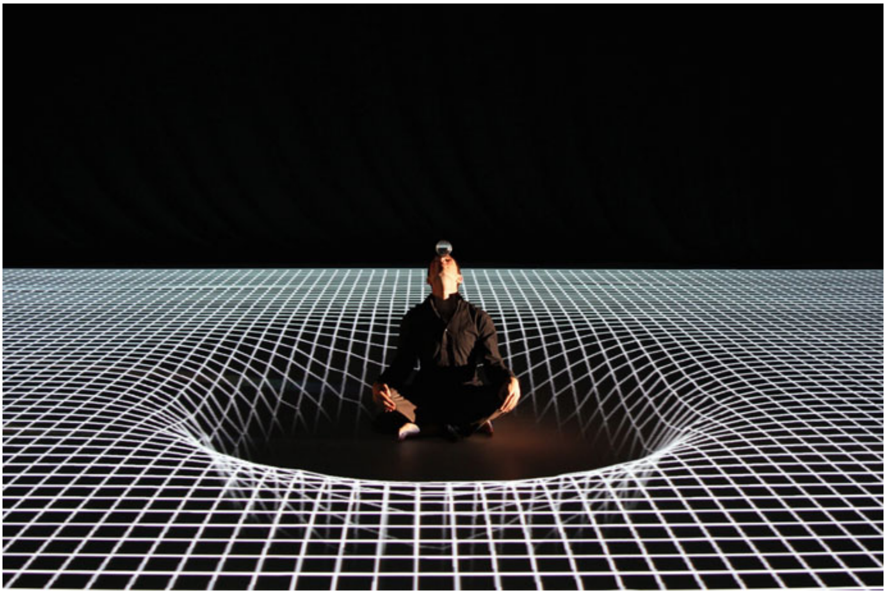

Photo from the show: *Cinématique* (2010) ©AMCB

Part VIII
Venice

The Ships, Shipyards and Ferries of Jacopo de' Barbari

Guglielmo Zanelli

Jacopo de' Barbari's bird's-eye perspective plan of Venice is a xylograph executed on six exceptionally large (282 cm long × 136 cm high) woodblocks, from which about 20 prints are know to exist (Fig. 1).

De' Barbari was a Venetian painter and above all engraver who lived between 1450 and 1516. Little is known about his life and artistic work. One of his best-known works, the portrait of Luca Pacioli now housed in the Capodimonte Museum in Naples, shows his interest in mathematics. What little biographical information there is attests to his activity in Italy and his journeys to the principal cities in Germany. He met Albrecht Dürer during the German artist's first say in Venice, and around 1497 began his collaboration with the wealthy German merchant Anton Kolb, who then published, in 1500, the exceptional view of the city of Venice.

The first thing we see looking at the plan as a whole is the dilatation of the part of the city known as the Castello district; it is almost as though the artist wished to emphasise the role of the Arsenale in sustaining the economic, commercial and maritime life of Venice. A second look reveals the absence of beaches and the sea, as if the ports providing access to the lagoon and the fortifications that protect them were a matter of national security.

Going further into the details of the view, the reading of it becomes even more interesting. We are struck by the figure of Mercury, who from on high dominates the entire city and sends a strong, positive message about Venice's commercial capabilities: *MERCURIUS PRE CETERIS HUIC FAUSTE EMPORIIS ILLUSTRO* (I, Mercury, shine favourably on this market which surpasses all others). Below appears another divinity, Neptune, lord of the seas, whose trident bears another strong affirmation: *AEQUORA TUENS PORTU RESIDEO HIC NEPTUNUS* (I, Neptune, live here keeping the waters in this port calm). In de' Barbari's depiction, the two gods have taken the place of the historic symbols of the Repubblica Serenissima, the lion of San Marco and the figure of Justice with the scales and sword. With Christopher Columbus's discovery of the new continent a few years earlier, Venetian commerce needed to relaunch itself internationally, and the new

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Fig. 1 *Perspective plan of the city of Venice,* Jacopo de' Barbari, woodcut of 6 woodblocks, 1500, Museo Storico Navale di Venezia (The creation of the present-day Museo Storico Navale di Venezia (Naval History Museum) is due to Giovanni Casoni, an engineer of the Royal Imperial Navy, who in 1825 assembled the collection of relics, weapons and artillery that had escaped the French sack of the Venice Arsenale. The initial collection was hosted in the Sale d'armi at the Ports, fittingly restored, and was enriched and added to over time. Casoni himself donated part of his library and the *Perspective plan of the city of Venice* by Jacopo de' Barbari. When Venice was united to the Savoy Kingdom of Italy, the Museo Marittimo of the Venezia Arsenale received objects and relics from all over Italy. In 1956 the entire collection was transferred and reorganised in the present-day location in the granaries of San Biagio. It is the most important museum of naval history in Italy.) (macrophotographic reproduction by the author)

metaphor introduced by the artist was a step in this 'modern' direction: in spite of the new Atlantic trade routes, the 'queen' of world maritime commerce at the dawn of the new century was still Venice. To underscore the economic activities of Venice's maritime empire, one part of the city is artificially enlarged: the area whose centre of gravity is the Arsenale (the state-owned maritime industry par excellence) and the private shipyards and armouries that surround it (Fig. 2).

What de' Barbari is proposing is a message of pacific prosperity and absolute industriousness. Jacopo, in depicting ships and watercraft in all parts of the city, is doing nothing less than making clear to the interested observer the security and protection that the waters of the city are capable of providing to ships, in terms of ease of docking and anchoring, as well as the intrinsic capacity, thanks to the presence of the Arsenale, to also assure protection and security for its commercial routes towards the Orient.

Rather than enter into all the questions, still unresolved, regarding how the surveys were made, what instruments were used to make them, and whether the execution of the view was done by a single person or by a collective, let us instead look at the ships and watercraft so carefully and accurately represented in the perspective view of Venice.

Fig. 2 Detail of Neptune, *Perspective plan of the city of Venice*, Jacopo de' Barbari, woodcut of six woodblocks, 1500, Museo Storico Navale di Venezia (macrophotographic reproduction by the author)

To reinforce the sense of the sea, which pervades Jacopo's entire view, the artist engraved in every part of the city and the lagoon an impressive number of ships, boats and small watercraft: there are almost 600 of them, of which 148 are large mercantile ships, 85 are freighters of lesser tonnage, and 300 are smaller boats. Along with the ships and boats are also depicted the shipyards that built them (called *squeri* in the Venetian dialect), the 'ports' where they can find safe harbour, and the water ferries that provide public transportation within the intricate urban fabric (Fig. 3).

What de' Barbari has shown are the ships that permitted Venice to create and expand her maritime empire: the *navi tonde* (round ships), large sailing ships for commerce, and the *navi sottili*, doubly-propelled by oars and sails, used to protect the trade routes.

In the view, we see *navi tonde* galore: from the well-known *cocche* (cogs) to the *marciliane* (a generic name for ships used to transport goods), all represented, as far as scale allows, with precision down to the smallest detail. The most famous Venetian merchant ship was the *cocca* (cog). Those shown anchored in the San Marco basin have up to three covered bridges and four masts on which both lateen and square sails could be raised. There were other kinds of merchant vessels in addition to cogs and *marciliane*, each with different names and uses: *barze, caracche, petacchi, siabecchi, tarette, trabacoli,* and so forth. De' Barbari completed the panorama of Venetian commercial watercraft with the depiction of the vessels used for trade along the Adriatic coast (*brazzere* and *tartane*), those used for rivers

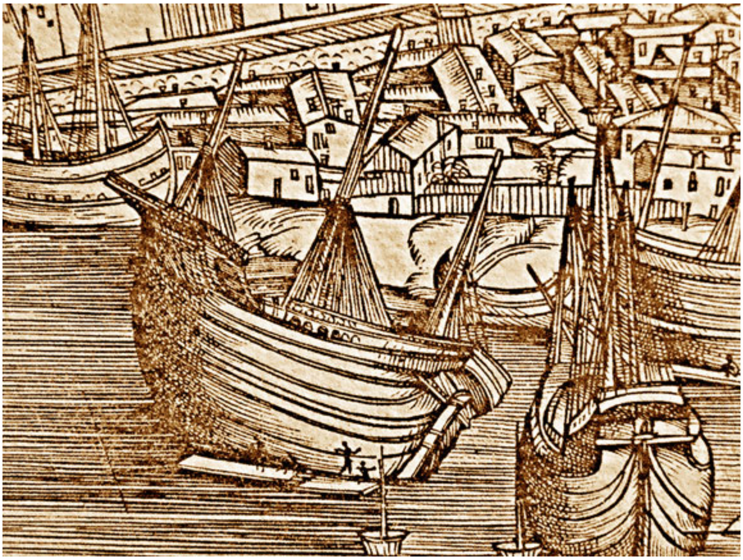

Fig. 3 Detail of a shipyard, *Perspective plan of the city of Venice*, Jacopo de' Barbari, woodcut of six woodblocks, 1500, Museo Storico Navale di Venezia (macrophotographic reproduction by the author)

(*rascone* and *batele*), and those used in the lagoon (*burci* and *peate*). Also seen are numerous boats used for fishing (*bragozzi* and *bragozzetti*) and for transporting people (*burchielli, barchete* and *gondole*).

The principal kind of *navi sottili* was the *galea* (galley, also called *galera*), adopted by all navies of the Mediterranean. Galleys were long a part of the history of this ancient sea, and accompanied the growth of Venice. There were two main types of galley: the *galea sottile*, prevalently used for military purposes, and the *galea grossa* or *galea da mercanzia*, used mainly for commerce.

Venetian commercial galleys were used in small teams called *mude*, along predetermined sea lanes. Such maritime routes covered the entire Mediterranean, and also went as far as the Atlantic to reach London and Bruges. A good part of Venice's commercial organisation was based on these sea lanes, at least until the end of the 1400s.

While the boats in the view may catch the observer's eye, the plan also draws his attention to everything that is closely related to ships, beginning with the ports that host them all the way to the places where they are constructed: the *squeri* (private shipyards) and the state-owned Arsenale.

Even though the entire lagoon is an immense and secure port, from De' Barbari's view we can see that at the beginning of the sixteenth century Rialto was still an important port for Venetian commerce. The presence of numerous *trabacoli*, *rascone, burchielli* and *peate* along the basements that surround the Rialto Bridge (a bridge that is shown still constructed of wood, with a central part that can be opened

Fig. 4 Detail of the Rialto Bridge. *Perspective plan of the city of Venice,* Jacopo de' Barbari, woodcut of six woodblocks, 1500. Museo Storico Navale di Venezia (macrophotographic reproduction by the author)

to allow for the passage of ships with masts) define the site as a natural terminal for the river trade with the Venetian and Padovan mainland (Fig. 4).

However, the largest and most important port of the city was the one located in the San Marco basin, characterised by the vast spaces that existed for anchoring the largest ships between the Grand Canal, the island of San Giorgio Maggiore and the church of San Domenico in the Castello district. The San Marco basin was thus *the* Venetian port par excellence: the circa 60 great vessels (mostly *cocche* and *marciliane*) that are depicted there, docked or anchored by shores and piers, provide a detailed and accurate testimony of its function. Other satellite ports, complementary but still important, were found in the San Pietro and Giudecca canals. These two ports were used for the docking, loading and unloading of boats used for fishing, as well as for the transportation of goods throughout the city.

Two other locations in the city acted as points of arrival, unloading and first treatment of a raw material that is essential for shipbuilding: timber. In the northern lagoon, near the Basilica of San Giovanni e Paolo, was situated the maritime terminal dedicated exclusively to the barges of timber that arrived to the Arsenale via the Piave River from the regions of Cadore and Cansiglio. In the southern lagoon arrive the barges coming from the Brenta and Adige Rivers, docking in the Dorsoduro district along the Giudecca Canal.

The areas adjacent to the ports teemed with the shipyards called *squeri* in the Venetian dialect, which De' Barbaro depicts in full activity. These Venetian shipyards were scattered throughout the city, with some even located along the

Grand Canal, between San Marco and San Vidal. However, the largest ones, called *squeri da nave* or *squeri da grosso*, which needed the greatest spaces and the deepest waters, were located along the city's southern periphery, on the shores facing the San Marco Basin and the Giudecca Canal, respectively in the Castello and Dorsoduro districts. The most important ones were found in the industrial zone of Castello, near the public *squeri* of the Arsenale, in the parish of S. Domenico in the bend between the churches of S. Biagio and S. Giuseppe. The way de' Barbari depicts these yards provides a precise idea of the kinds of ships worked on there: *burci, peate* and *barchete* were built and repaired on the Grand Canal and in the smaller inner canals of the city.

As mentioned earlier, de' Barbari devoted a significant portion of the plan to a view of the Arsenale, that is, the area of the state-owned shipyards for galleys, showing the *Cuore dello Stato Veneto* (Heart of the Venetan State, as the great shipyard was called in the seventeenth century) as an added asset to the port of Venice. Because of the importance given to the complex in the view, Jacopo's representation of the Arsenale (within which there were between 1,000 and 3,000 smaller shipyards) is a valuable source of details regarding the organisation of the work. We can see good depictions of the two oldest parts of the Arsenale, the *Arsenale Vecchio* (to the west) and the *Arsenale Nuovo* (to the east); north of the *Arsenale Nuovo* can be seen the perimeter walls and the wet dock (still under construction) of the *Arsenale Nuovissimo*, which would not be completed until the second half of the 1500s (Fig. 5).

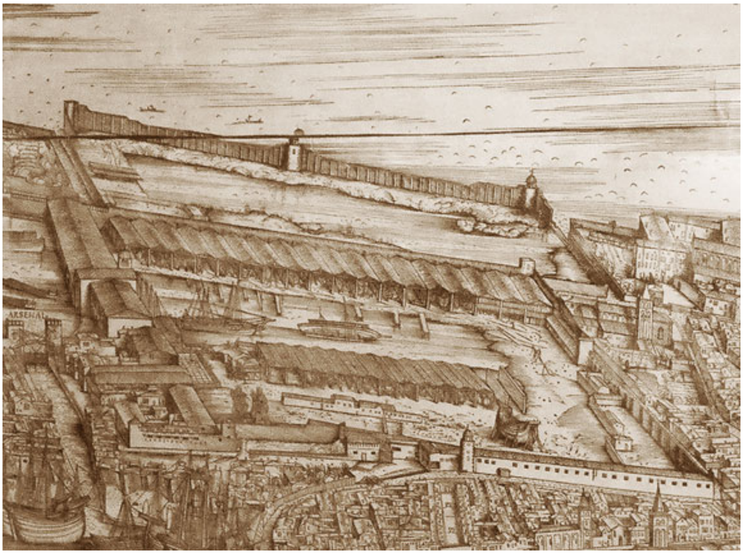

Fig. 5 Detail of Arsenal, *Perspective plan of the city of Venice,* Jacopo de' Barbari, woodcut of six woodblocks, 1500. Museo Storico Navale di Venezia (macrophotographic reproduction by the author)

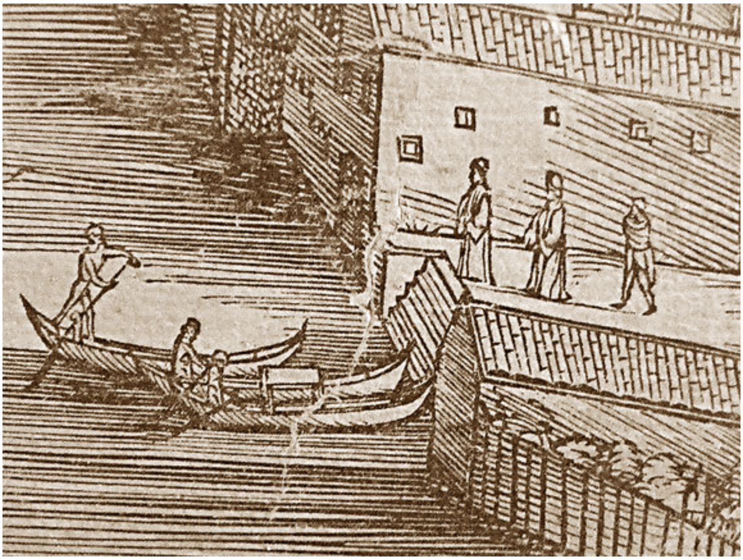

Fig. 6 Detail of the ferry of the island of S. Giorgio Maggiore. *Perspective plan of the city of Venice,* Jacopo de' Barbari, woodcut of six woodblocks, 1500. Museo Storico Navale di Venezia (macrophotographic reproduction by the author)

The woodcut seems to be an endless font of information. Noteworthy among all it shows is the means of public transportation for residents and visitors, the so-called *traghetti* (ferries). In Venice the term *traghetto* indicates a vessel used to cross large and small canals; there were *traghetti di dentro* (inside the city) and *traghetti di fora* (used to connect Venice with the terra firma and neighbouring states).

The *traghetti*, both before and after the building of bridges, were the most important element of the urban infrastructure for Venice's economic and social systems: thanks to them mobility in and around the city were insured for residents and foreigners and for their goods. We might ask what use it was to have ferries close to bridges (there were at least ten ferries around the Rialto Bridge), but it should be recalled that, in addition to taking passengers from one side to the other, the ferry boats also made it possible to reach any island in the lagoon and the cities on the terra firma that had rivers and canals. Ferries, according to their importance, had licenses that gave them greater or lesser range of travel (Fig. 6).

In each canal are rendered perfectly legible the numerous *stazi* (places for stopping and anchoring) destined for the smaller *traghetti*, watercraft used to transport people and things. In the view, as far as the perspective allows it, we can see the stops of about 20 *traghetti di dentro*, and along the canals we find, like today, caravans of small boats and gondolas destined, in all likelihood, for tourists. In the port areas, as mentioned earlier, we find instead the stops for the *traghetti de fora* destined for the transportation of goods, such as those in the canal S. Giorgio and the Giudecca used for the *rascone* for the transportation of hay.

The wealth of detail offered by de' Barbari's (and his possible collaborators) woodcut confirms, apart from the obvious artistic success, both the commitment of the author to its conception and execution, and the importance of the work as a document of exceptional quality in terms of information. Onto the idyllic image of the city created by the ships and boats that cross its calm waters is powerfully superimposed that of its commercial and maritime vocation, which is exalted and projected beyond the invisible shores by an infinity of naval elements and details that are exemplified and subtended by the ships and boats. The heads that represent the eight principal winds (complete with the names of the minor winds) are not only a valuable aesthetic enrichment of the work, but also provide, thanks to the accuracy of their orientation (the use of a compass appears evident) a scientific guarantee for the more massive and immediate presence of Mercury and Neptune, tutelary deities of the city. The message that de' Barbari wished to convey is clear: 'Venice, its rulers and its ships can be trusted'.

Translated from the Italian by Kim Williams

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